ben@math.uh.edu PreCalculus 1330 www.math.uh.edu/~ben

Turn your homework in BEFORE the start of class. If you turn it in with the poppers, it is NOT graded.

Make sure you bubble the section number, assignment number, your people soft number to receive credit for your poppers.

Watch the dates and times for quizzes.

~~Office hours in CASA – 1 4 Monday and Wednesday~~

Be considerate of those around you.

Unit Circle Trigonometry

§ 4.3

$$x^2 + y^2 = 1$$

Def

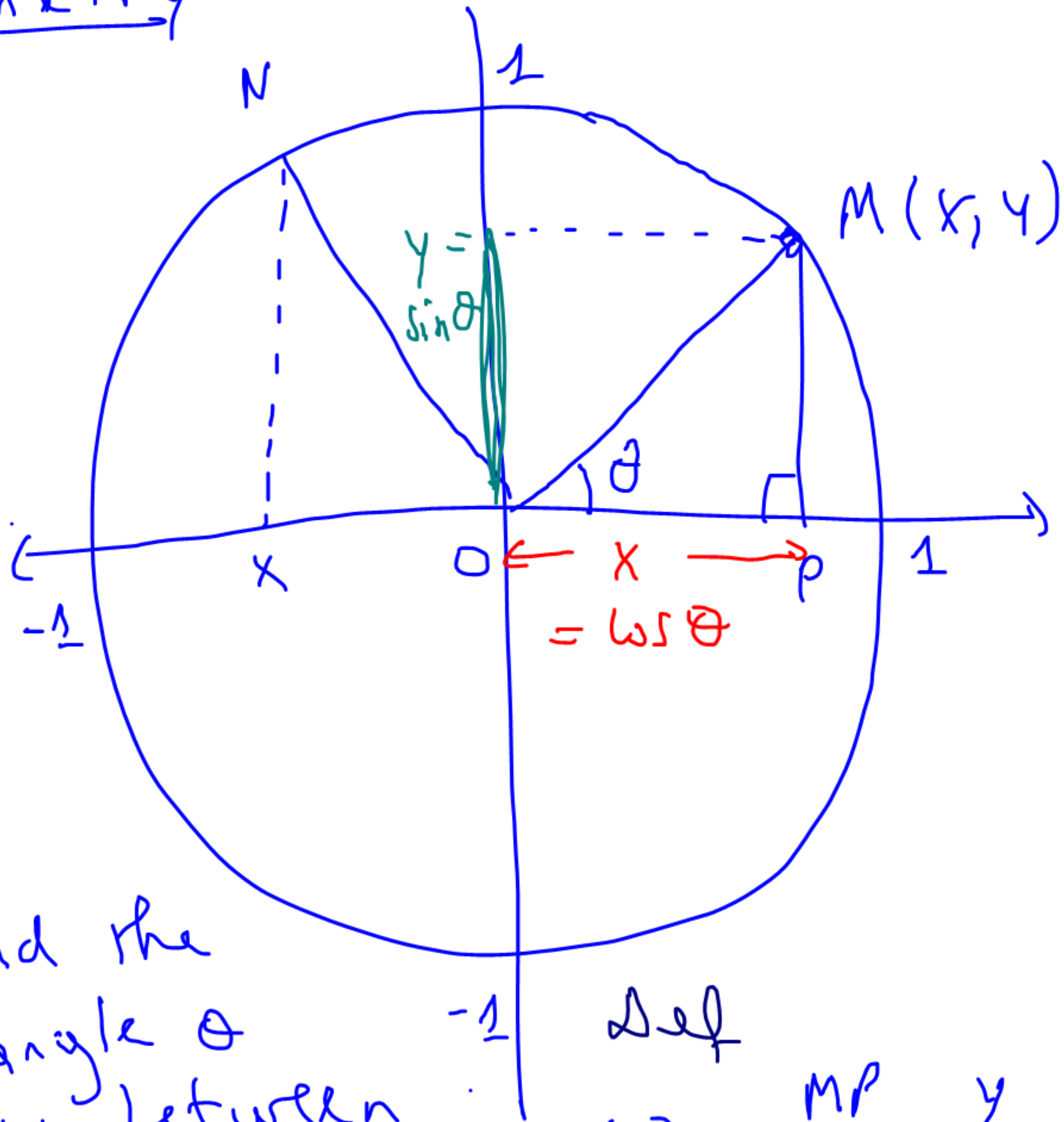
$$\cos\theta = \frac{OP}{OM} = \frac{x}{1} = x$$

$$\boxed{\cos\theta = x}$$

$$-1 \le x \le 1$$

$$\boxed{-1 \le \cos\theta \le 1}$$

Read the cosine of an angle θ on the x axis between -1 and 1



Def

$$\sin\theta = \frac{MP}{OM} = \frac{y}{1}$$

$$\boxed{\sin\theta = y}$$

$$\boxed{-1 \leq \sin\theta \leq 1}$$

Read the sine on the y axis between -1 and 1

<u>Def</u>

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x} ;$$

$$-\infty < \tan\theta < \infty$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{1}{\tan\theta} = \frac{x}{y} ; \quad -\infty < \cot\theta < \infty$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{1}{x} ; \quad \sec\theta \leq -1 \ \underline{\text{or}} \ \sec\theta \geq 1$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{1}{y} ; \quad \csc\theta \leq -1 \ \underline{\text{or}} \ \csc\theta \geq 1$$

First (and most important) trigonometric identity

$$x^2 + y^2 = 1$$ (unit circle; center $(0,0)$ radius $R = 1$)

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$ for any angle θ

$$\boxed{\cos^2\theta + \sin^2\theta = 1}$$

Ex

$$\cos^2 71^\circ + \sin^2 71^\circ = 1$$

$$\cos^2(2.5\,\text{rad}) + \sin^2(2.5\,\text{rad}) = 1$$

$$\cos^2(3x) + \sin^2(3x) = 1$$

$$\cos^2\theta + \sin^2\theta = 1$$

$$\cos^2\theta = 1 - \sin^2\theta = (1-\sin\theta)(1+\sin\theta)$$

$$\sin^2\theta = 1 - \cos^2\theta = (1-\cos\theta)(1+\cos\theta)$$

Quad II

$y > 0$ $\sin\beta > 0$

$x < 0$ $\cos\beta < 0$

$\tan\beta < 0$

$\cot\beta < 0$

$\csc\beta > 0$

$\sec\beta < 0$

Quadrant I

$y > 0$ $\sin\theta > 0$

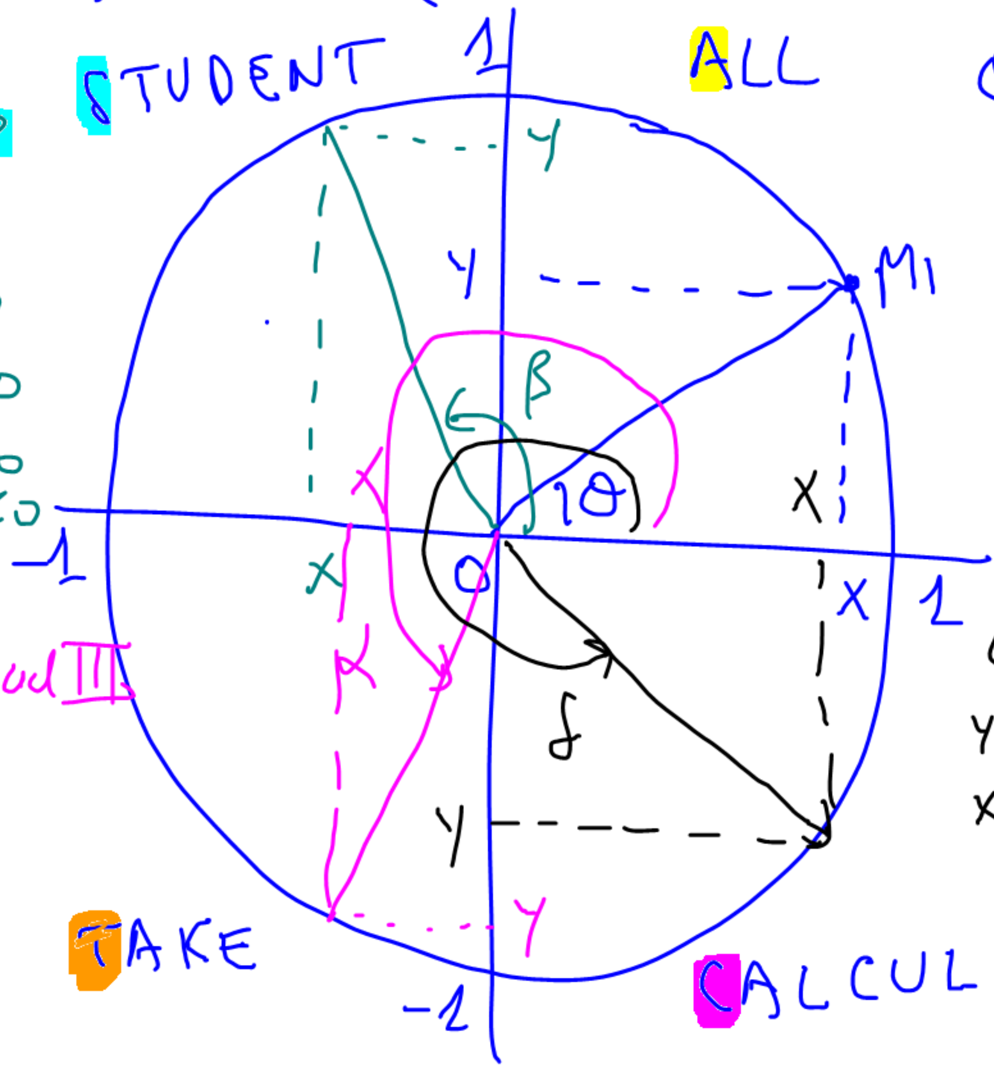
$x > 0$ $\cos\theta > 0$

$\tan\theta > 0$

$\cot\theta > 0$

$\csc\theta > 0$

$\sec\theta > 0$



Quad III

$y < 0$ $\sin\alpha < 0$

$x < 0$ $\cos\alpha < 0$

$\tan\alpha > 0$

$\cot\alpha > 0$

$\csc\alpha < 0$

$\sec\alpha < 0$

Quad IV

$y < 0$ $\sin\delta < 0$

$x > 0$ $\cos\delta > 0$

$\tan\delta < 0$

$\cot\delta < 0$

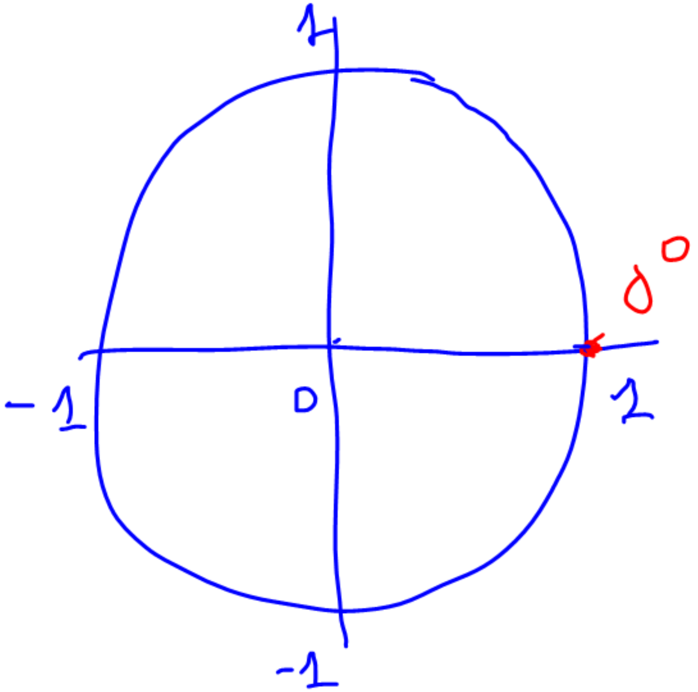
$\csc\delta < 0$

$\sec\delta > 0$

Trigonometric functions of fundamental angles.

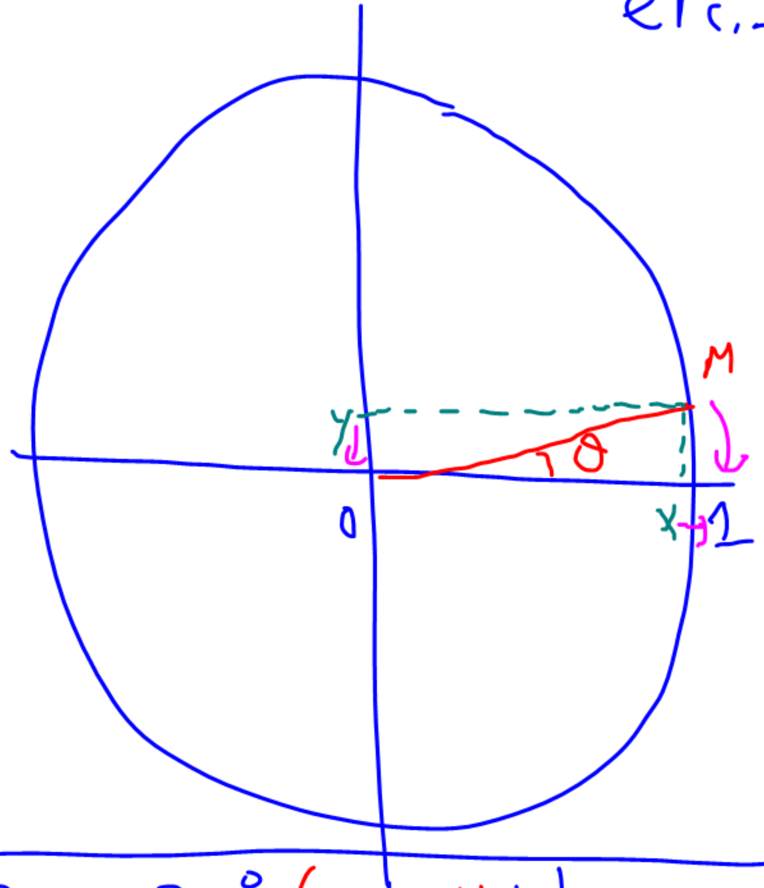
($0°$, $30°$, $45°$, $60°$, $90°$, etc...)

$\theta = 0°$ (or 0 rad)



$\sin 0° = 0$

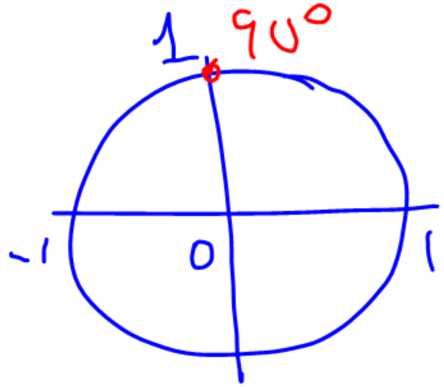
$\cos 0° = 1$



$\theta = 90°$ $\left(\frac{\pi}{2}\right)$

$\sin 90° = 1$

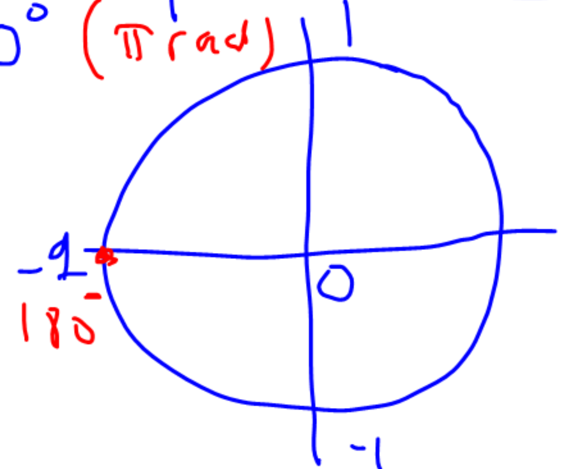
$\cos 90° = 0$

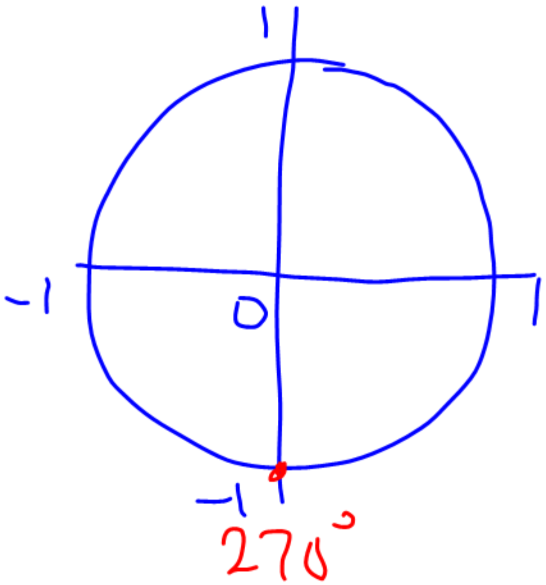


$\theta = 180°$ (π rad)

$\sin 180° = 0$

$\cos 180° = -1$



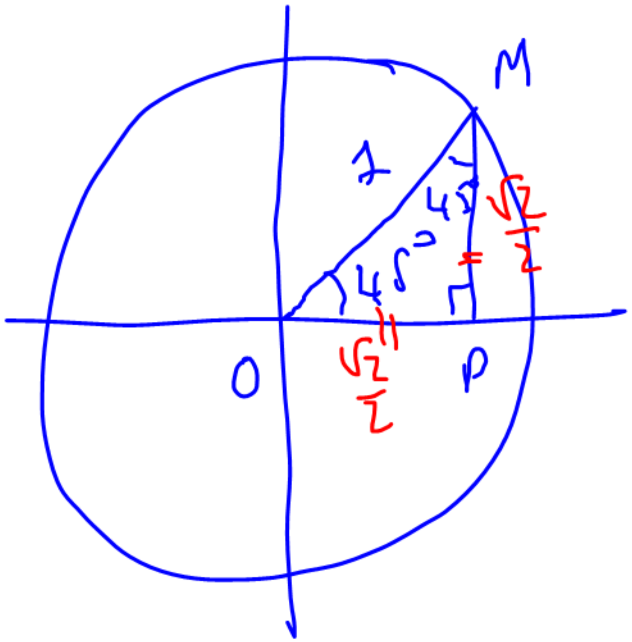


$\theta = 270°$

(or $\frac{3\pi}{2}$ rad)

$\sin 270° = -1$

$\cos 270° = 0$



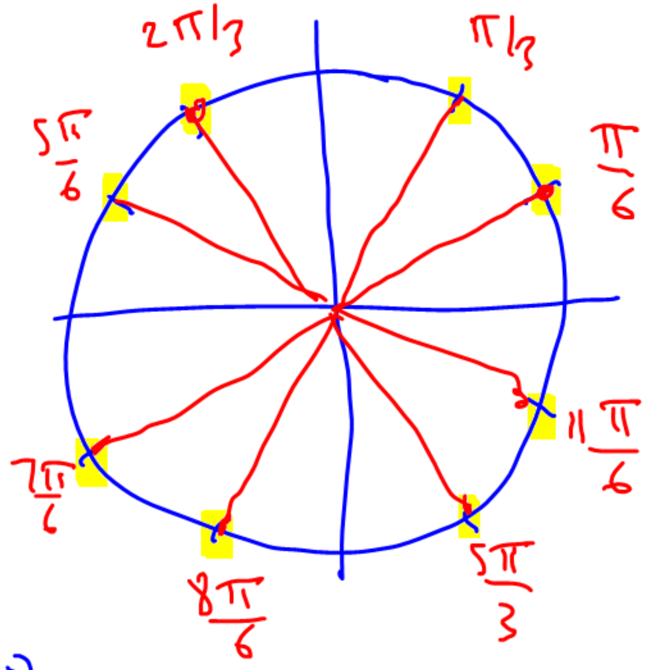
OMP is a $90° - 45° - 45°$

$\sqrt{2} - 1 - 1$

$1 - \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}$

$$\boxed{\begin{gathered}\sin 45° = \frac{\sqrt{2}}{2} \\ \cos 45° = \frac{\sqrt{2}}{2}\end{gathered}}$$

For Non quadrantal multiples of 30°
$(0^\circ, 90^\circ, 180^\circ, \ldots)$

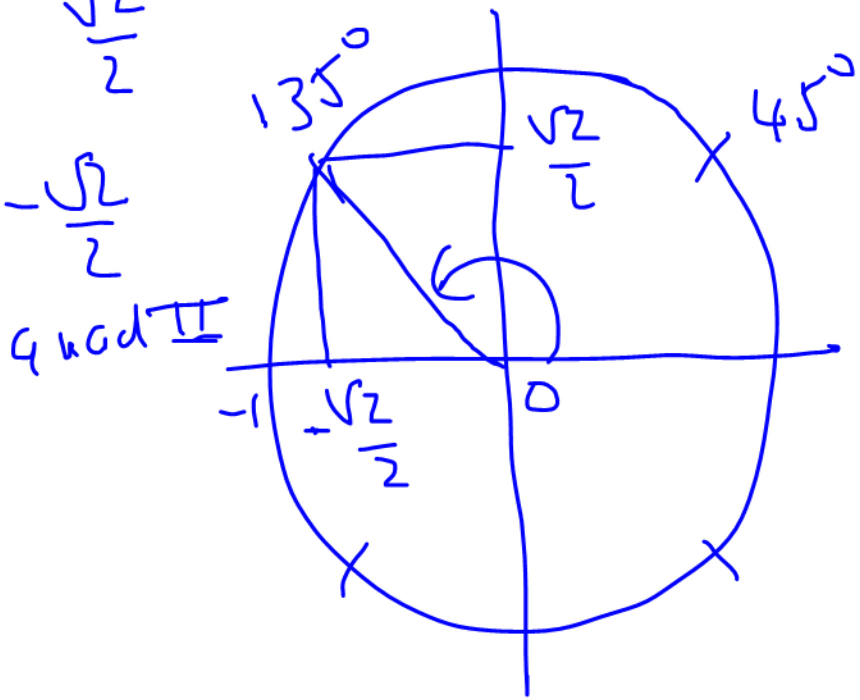


For the angles in yellow the sine and the cosine are always $\frac{\sqrt{2}}{2}$ within a sign.

Ex

$\sin 135^\circ = \frac{\sqrt{2}}{2}$

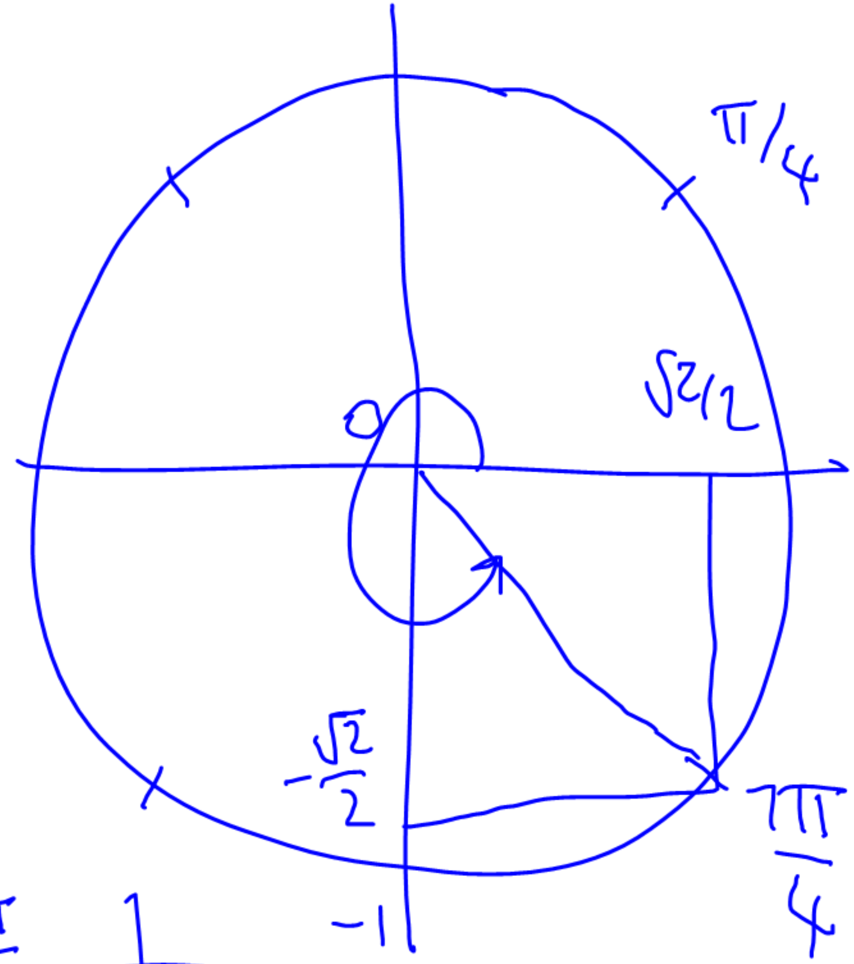
$\cos 135^\circ = -\frac{\sqrt{2}}{2}$



Ex

$$\sin\frac{7\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\cos\frac{7\pi}{4} = \frac{\sqrt{2}}{2}$$

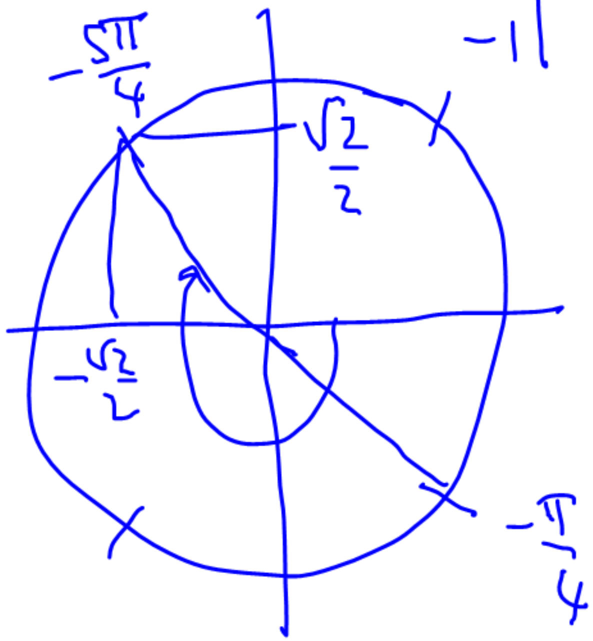


quad IV

Ex

$$\sin\left(-\frac{5\pi}{4}\right) = \frac{\sqrt{2}}{2}$$

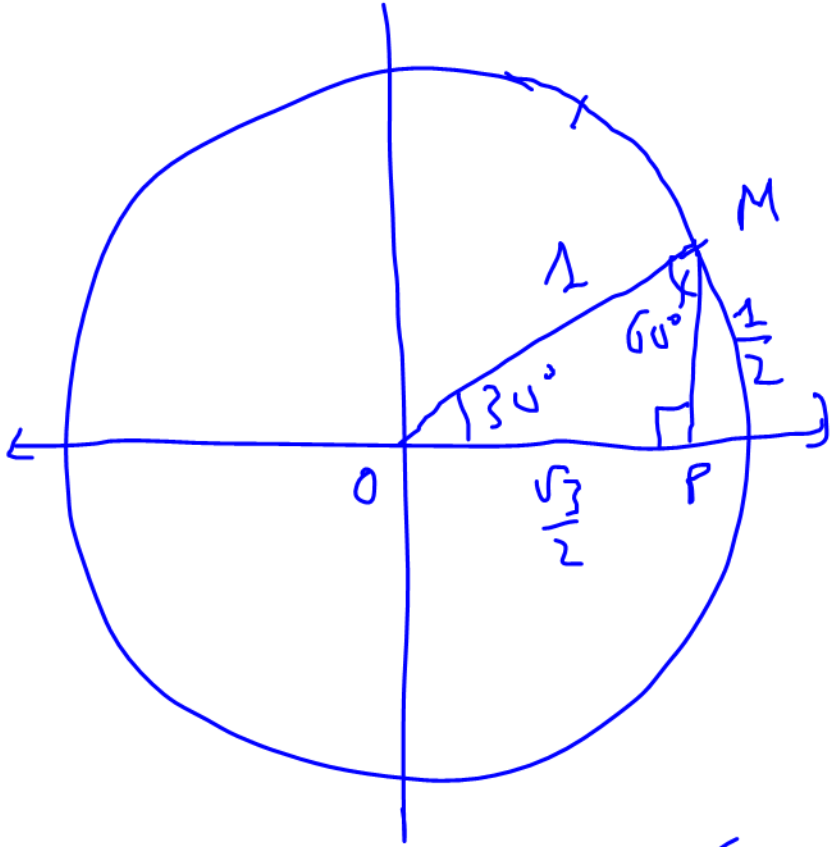
$$\cos\left(-\frac{5\pi}{4}\right) = -\frac{\sqrt{2}}{2}$$



$\theta = 30^\circ \quad \left(\frac{\pi}{6}\right)$

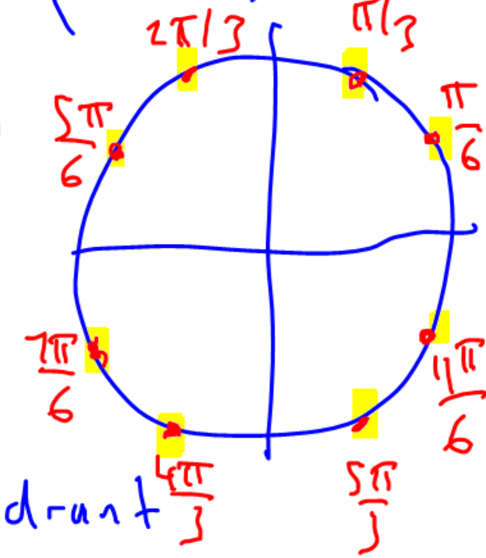
ΔOPM is a $90^\circ - 60^\circ - 30^\circ$

$1 - \frac{\sqrt{3}}{2} - \frac{1}{2}$



$$\boxed{\begin{array}{l} \sin 30^\circ = \frac{1}{2} \\ \\ \cos 30^\circ = \frac{\sqrt{3}}{2} \end{array}}$$

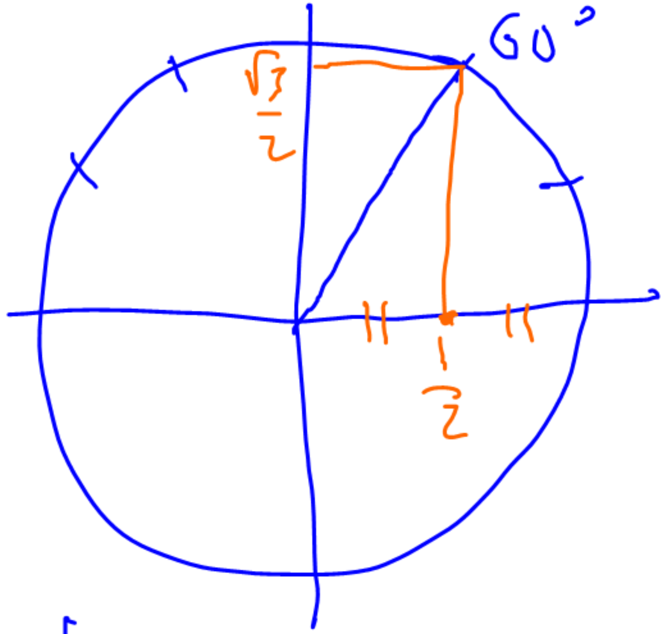
Rule For non quadrantal multiples of 30° (or 60°)
the sine and the cosine are always worth
$\frac{1}{2}$ and $\frac{\sqrt{3}}{2}$ within a sign
or the other way
the signs are determined by the quadrant



Ex

$\sin 60^\circ = \frac{\sqrt{3}}{2}$

$\cos 60^\circ = \frac{1}{2}$



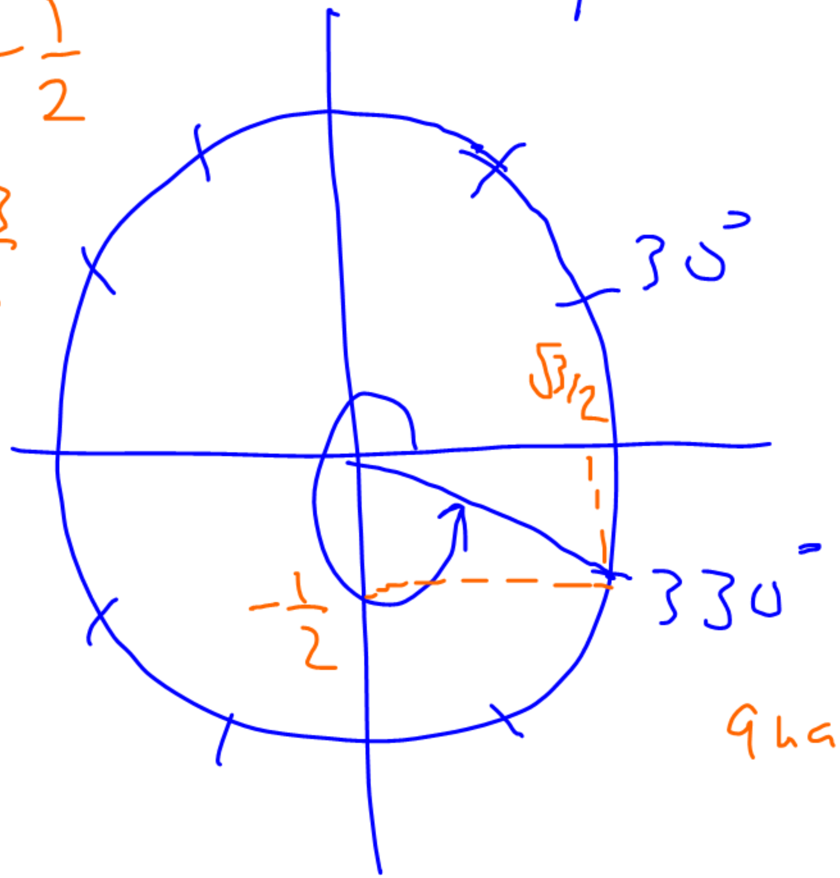
$\frac{\sqrt{3}}{2} \approx 0.85$

$\frac{1}{2} = 0.5$

Ex

$\sin 330^\circ = -\frac{1}{2}$

$\cos 330^\circ = \frac{\sqrt{3}}{2}$

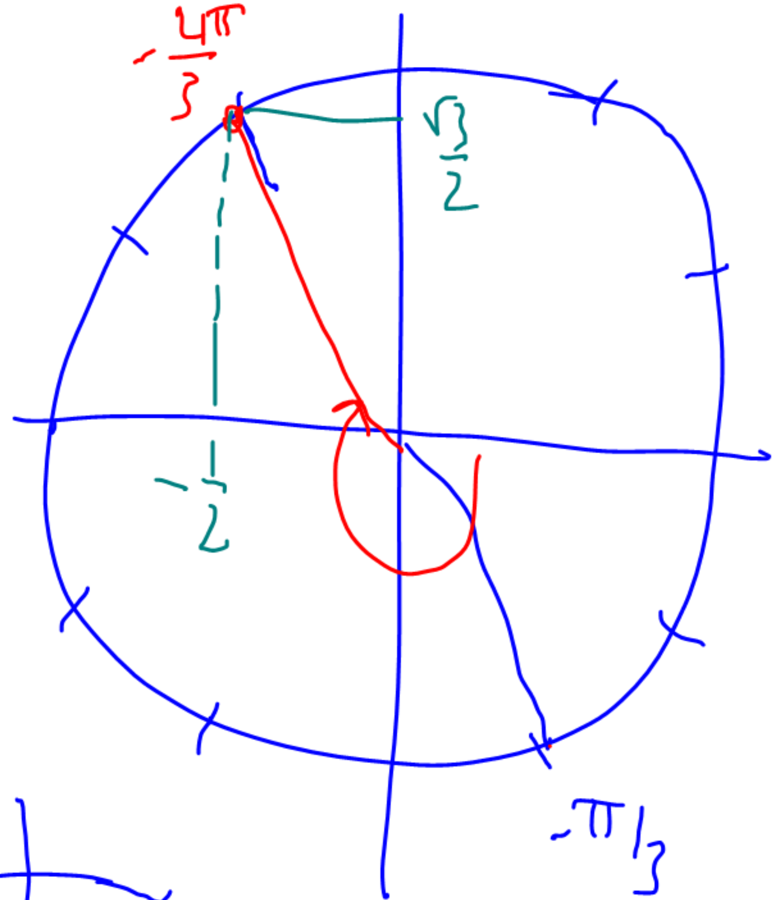


quad IV

Ex

$$\sin\left(-\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2}$$

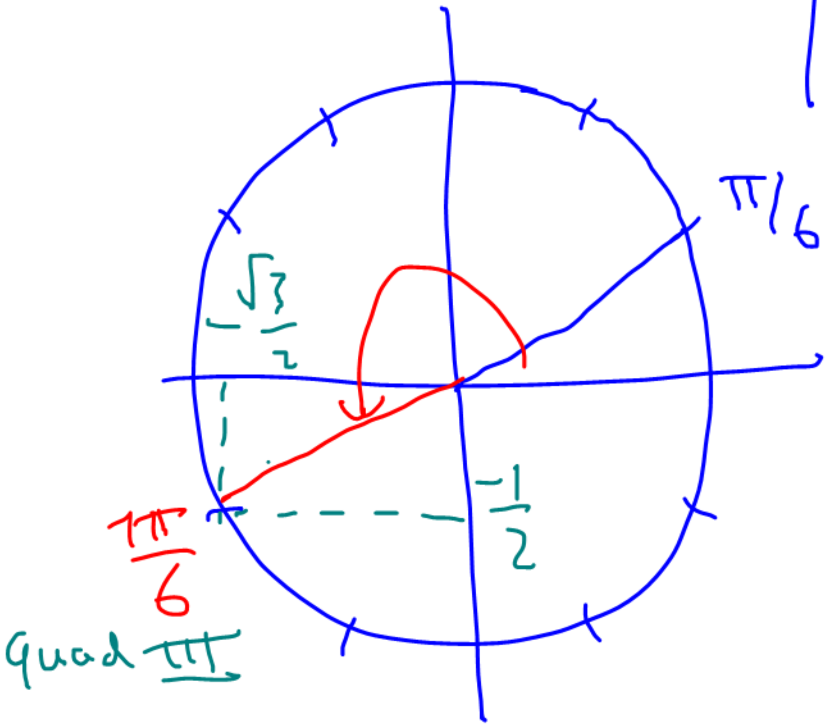
$$\cos\left(-\frac{4\pi}{3}\right) = -\frac{1}{2}$$

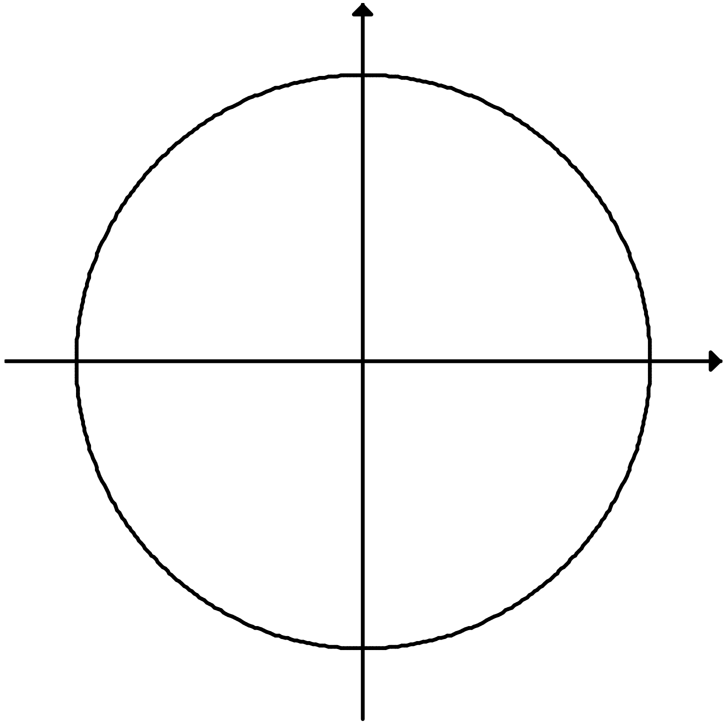


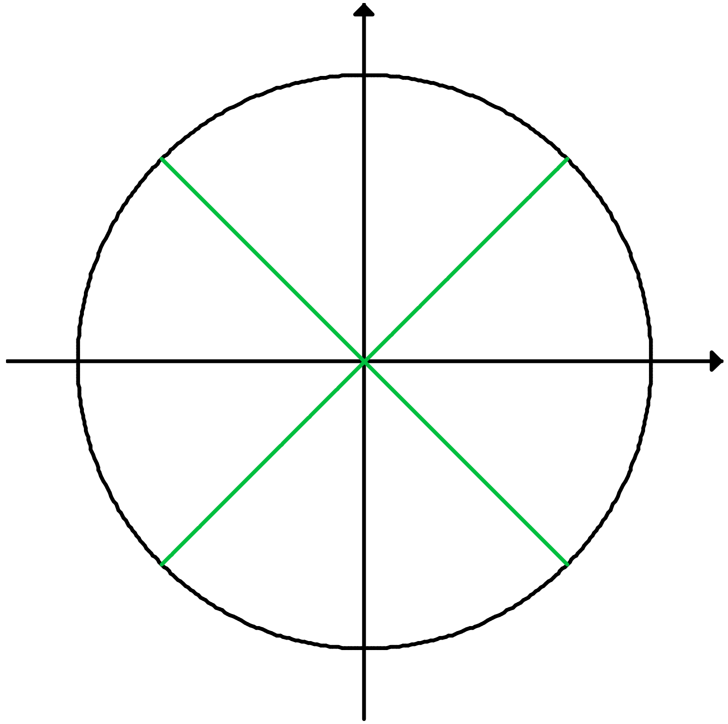
Ex

$$\sin \frac{7\pi}{6} = -\frac{1}{2}$$

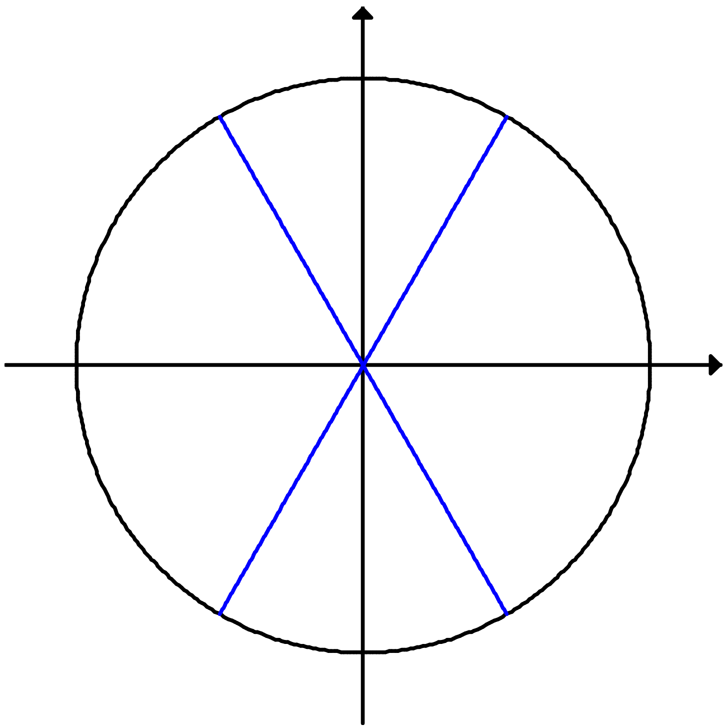
$$\cos \frac{7\pi}{6} = -\frac{\sqrt{3}}{2}$$

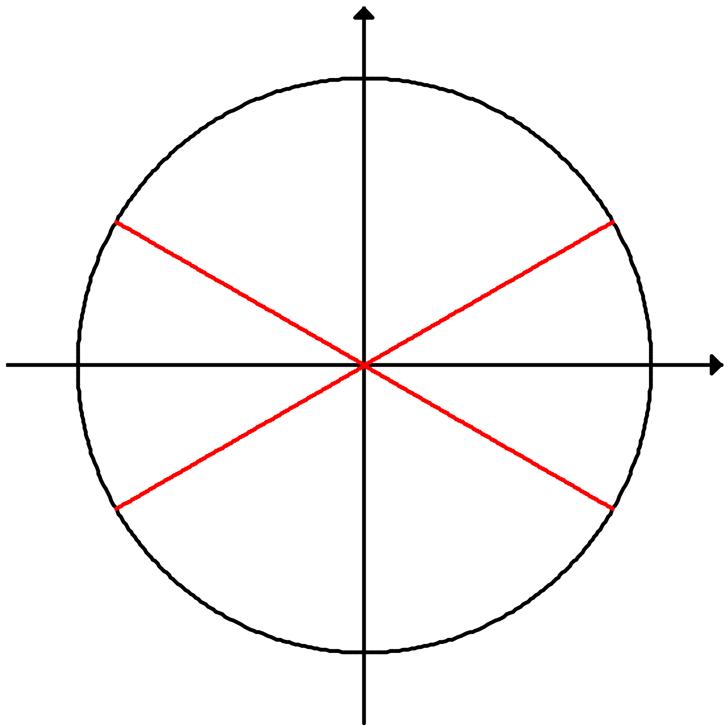






Unit Circle: Radius = 1

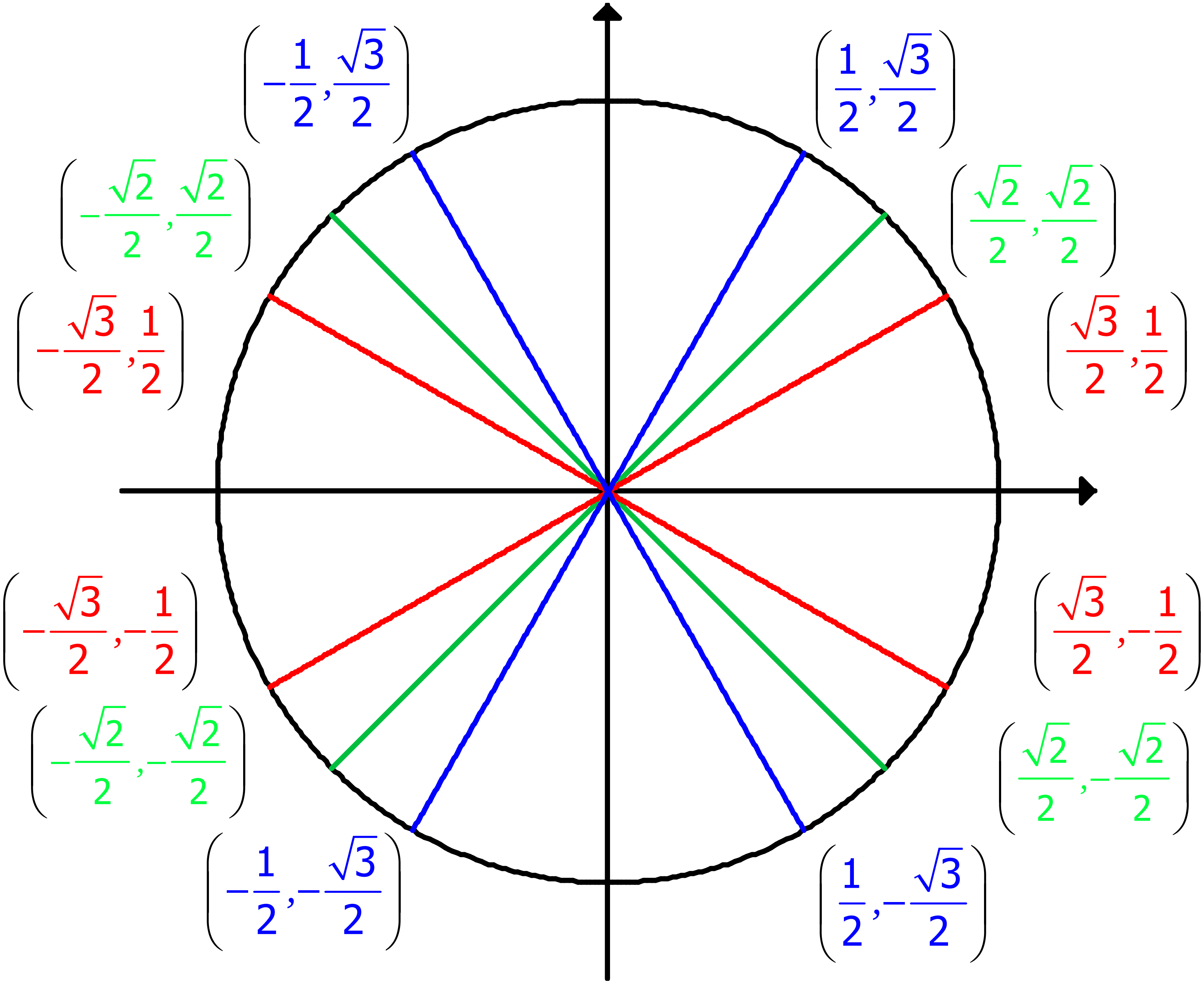




$30° = \frac{\pi}{6}$

$45° = \frac{\pi}{4}$

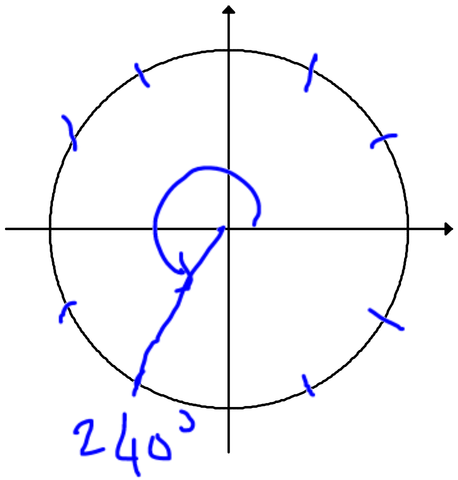
$60° = \frac{\pi}{3}$



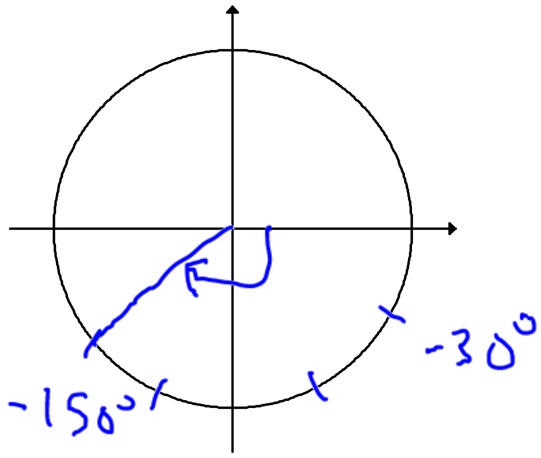
An angle is in standard position if its vertex is at the origin and its initial side is along the positive x axis. Positive angles are measured counterclockwise from the initial side. Negative angles are measured clockwise. We will typically use θ to denote an angle.

<u>Example 1:</u> Draw each angle in standard position.

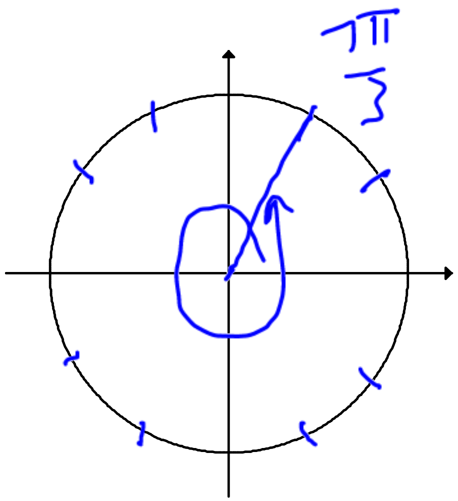
a. $240°$



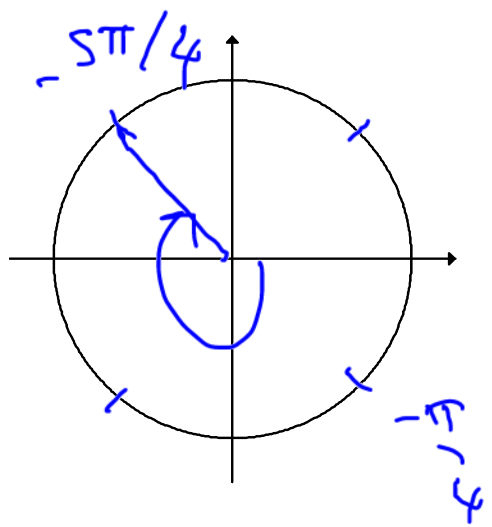
b. $-150°$



c. $\frac{7\pi}{3}$



d. $\frac{-5\pi}{4}$



Angles that have the same terminal side are called co-terminal angles. Measures of co-terminal angles differ by a multiple of 360° if measured in degrees or by multiple of 2π if measured in radians.

Example 2: Find three angles, two positive and one negative that are co-terminal with each angle.

a. 512°

$512° + 360° = \boxed{872°}$

$872° + 360° = \boxed{1232°}$

$512° - 360° = \boxed{152°}$

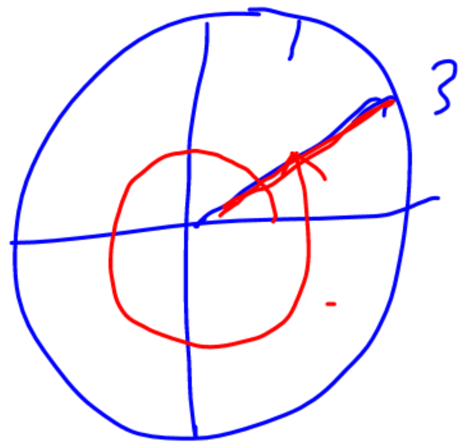
$152 - 360° = \boxed{-208°}$

b. $\frac{-15\pi}{8}$

$-\frac{15\pi}{8} + \frac{2\pi}{1}\frac{(8)}{(8)} = \boxed{\frac{\pi}{8}}$

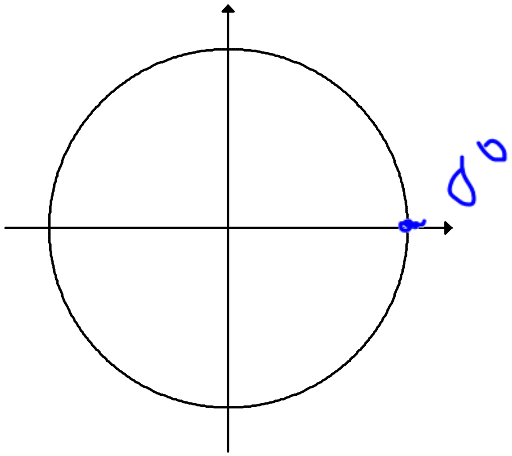
$\frac{\pi}{8} + 2\pi = \boxed{\frac{17\pi}{8}}$

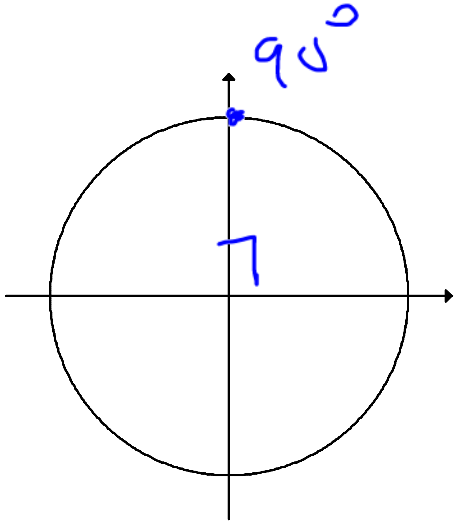
$-\frac{15\pi}{8} - 2\pi = \boxed{-\frac{31\pi}{8}}$

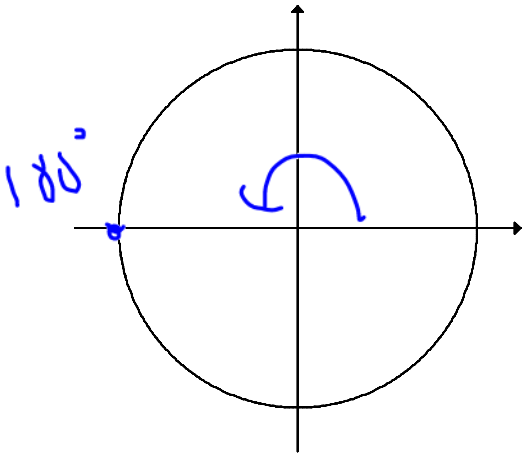


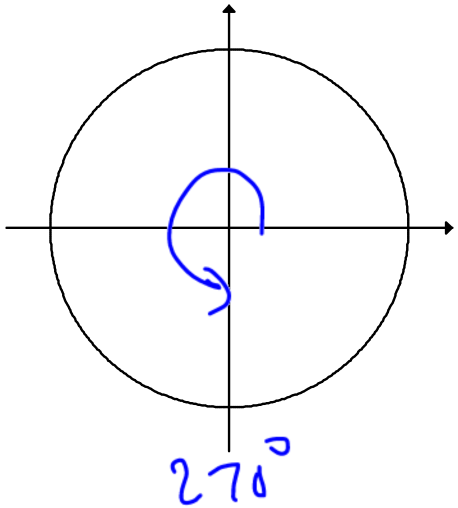
35° or 390°

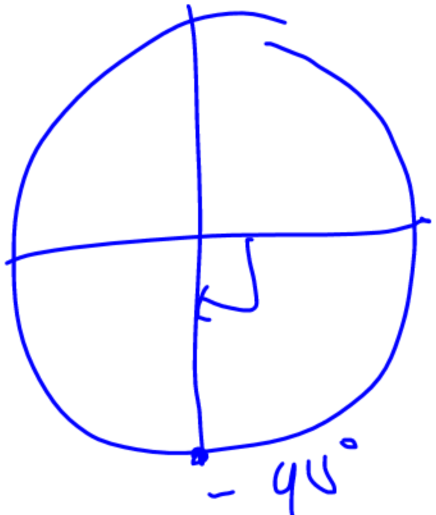
If an angle is in standard position and its terminal side lies along the x or y axis, then we call it a quadrantal angle.











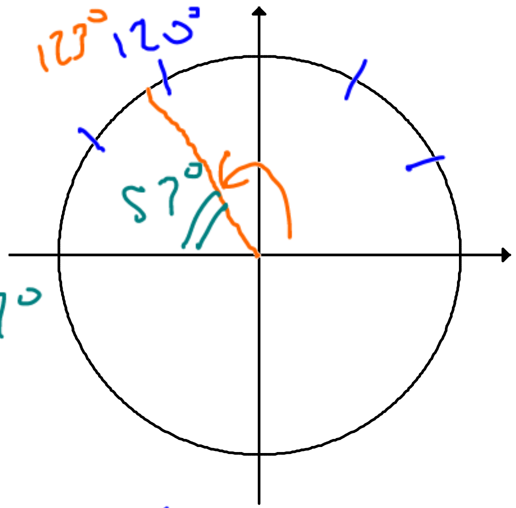
Suppose θ is angle in standard position and θ is not a quadrantal angle. The reference angle for θ is an acute angle of positive measure that is formed by the terminal side of the angle and the x axis. $0° < m\angle$ reference angle $< 90°$

"Drop" a perpendicular from the terminal side of the angle to the x-axis.

Example 3: Find the reference angles for each of these angles.

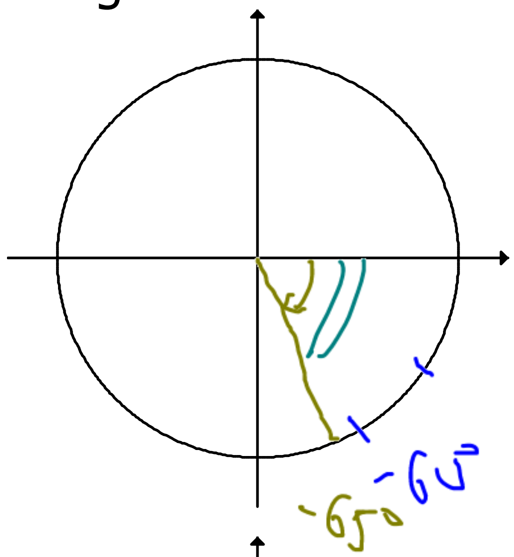
a. 123°

Reference angle
$180° - 123° = 57°$



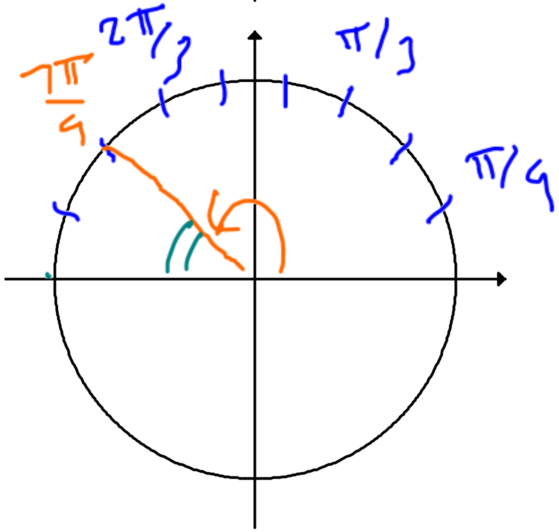
b. −65°

Reference
65°



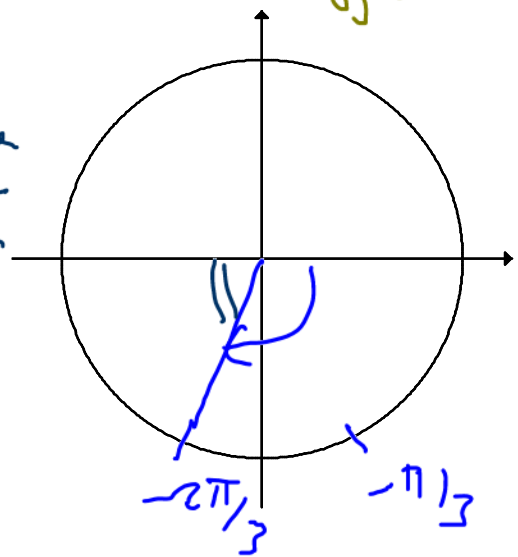
c. $\dfrac{7\pi}{9}$

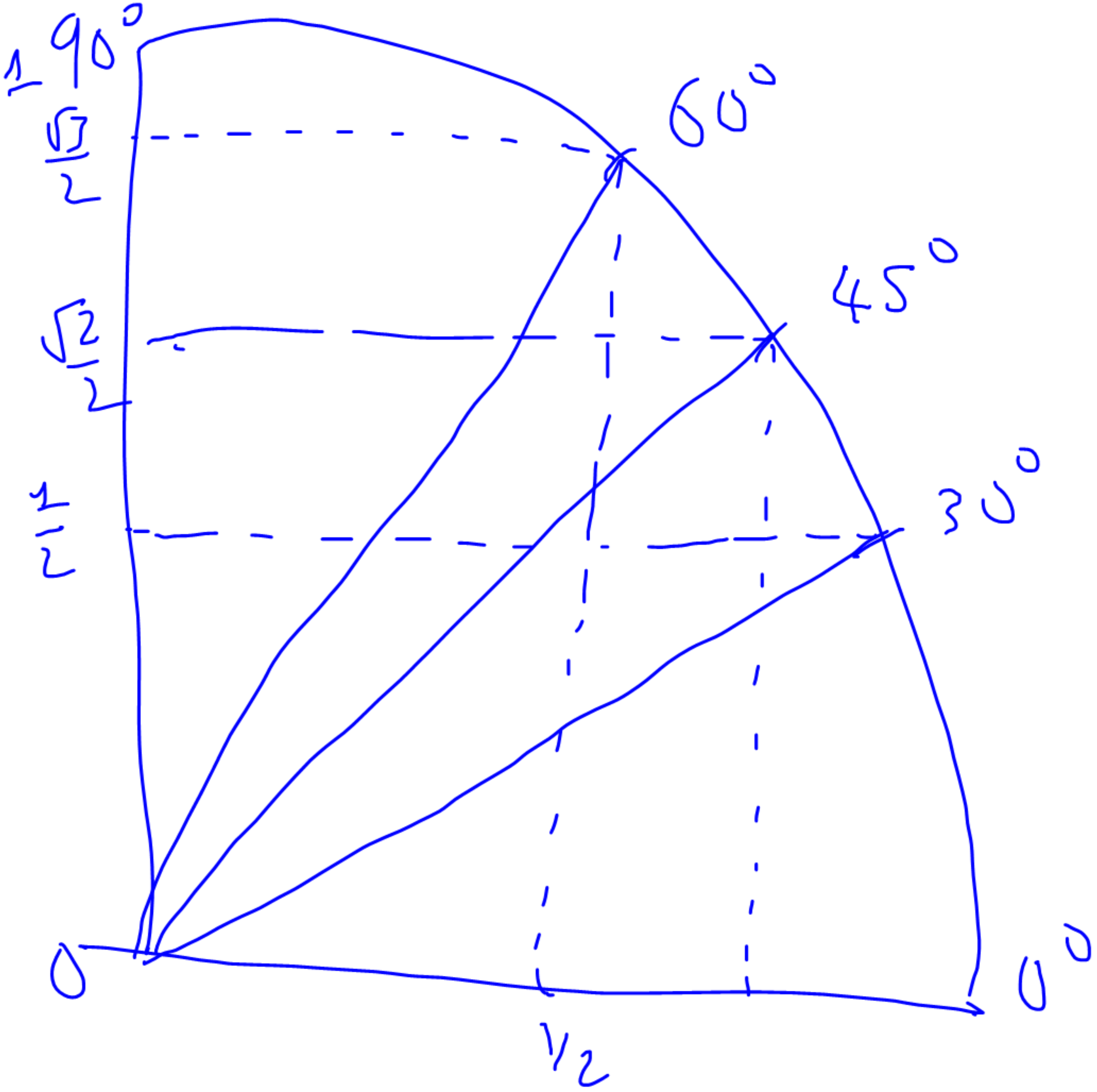
Reference angle $\dfrac{2\pi}{9}$



d. $\dfrac{-2\pi}{3}$

Reference angle $\dfrac{\pi}{3}$





$$\sin 0^\circ = 0 = \frac{\sqrt{0}}{2}$$

$$\sin 30^\circ = \frac{1}{2} = \frac{\sqrt{1}}{2}$$

$$\sin 45^\circ = \frac{\sqrt{2}}{2} = \frac{\sqrt{2}}{2}$$

$$\sin 60^\circ = \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$

$$\sin 90^\circ = 1 = \frac{\sqrt{4}}{2}$$

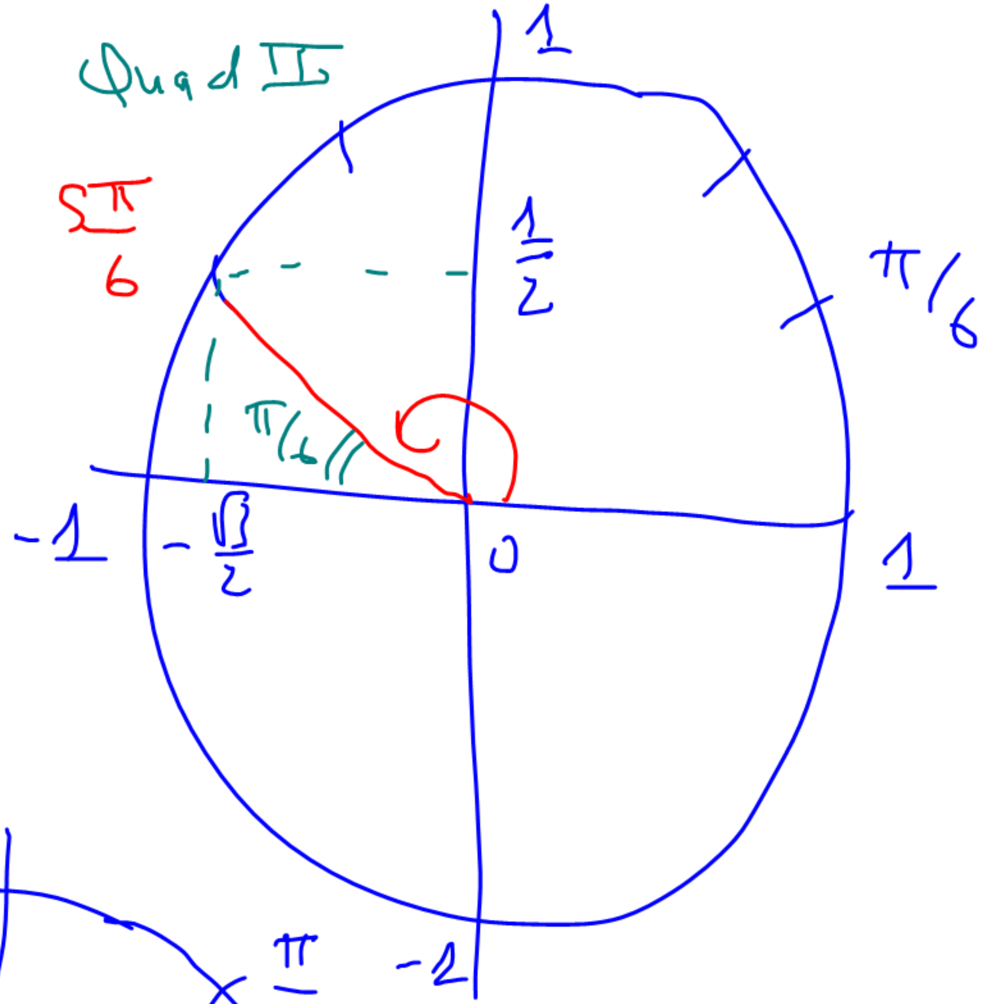
θ	0° (0)	30° $\left(\frac{\pi}{6}\right)$	45° $\left(\frac{\pi}{4}\right)$	60° $\left(\frac{\pi}{3}\right)$	90° $\left(\frac{\pi}{2}\right)$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	undef
$\cot\theta$	undef	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$	0
$\csc\theta$	undef	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	1
$\sec\theta$	1	$\frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$	$\sqrt{2}$	2	undef

Ex

Ref angle $\frac{\pi}{6}$

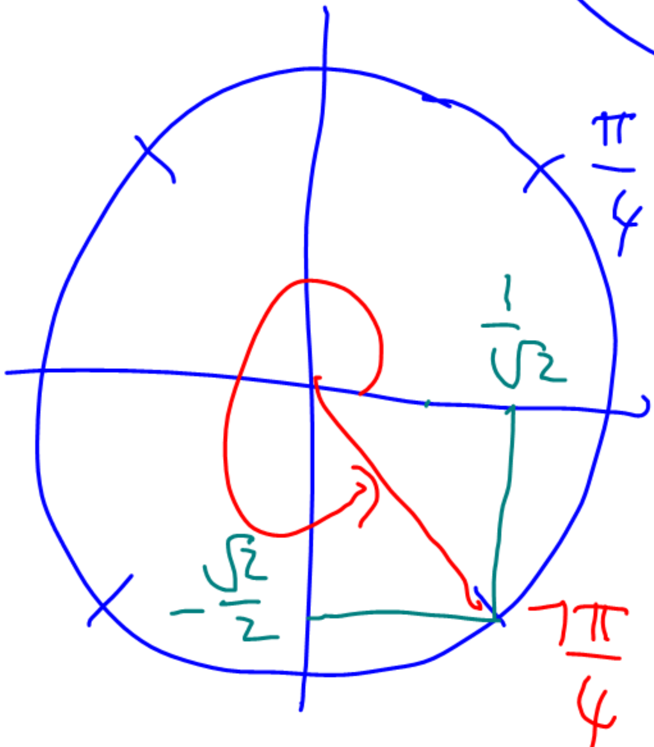
$$\sin\frac{5\pi}{6} = \frac{1}{2}$$

$$\cos\frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$$



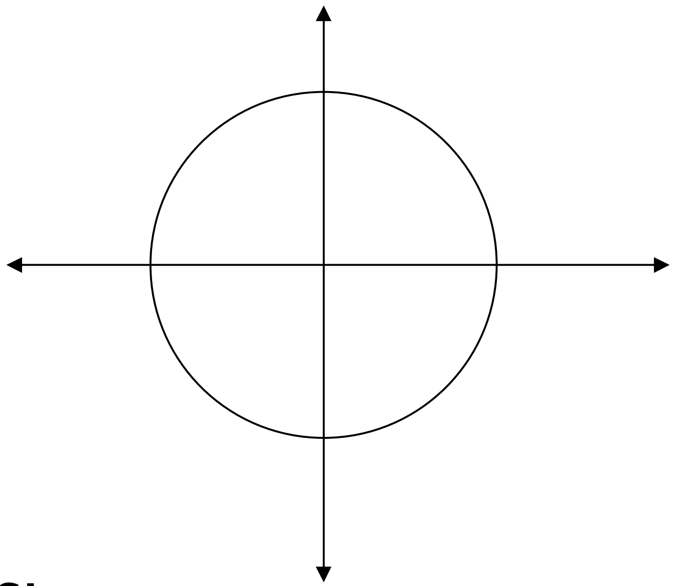
Ex

$$\sec\frac{7\pi}{4} = \boxed{\sqrt{2}}$$



Quad IV

We previously defined the six trigonometry functions of an angle as ratios of the length of the sides of a right triangle. Now we will look at them using a circle centered at the origin in the coordinate plane. This circle will have the equation $x^2 + y^2 = r^2$. If we select a point P(x, y) on the circle and draw a ray from the origin though the point, we have created an angle in standard position.



The six trig functions of θ are defined as follows:

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y} \quad (y \neq 0)$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x} \quad (x \neq 0)$$

$$\tan\theta = \frac{y}{x} \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (y \neq 0)$$

We will most often work with a unit circle with radius 1. In this case, each value of r is 1. This adjusts the trig functions as follows:

Trigonometric Functions of Angles

$$\sin\theta = y$$

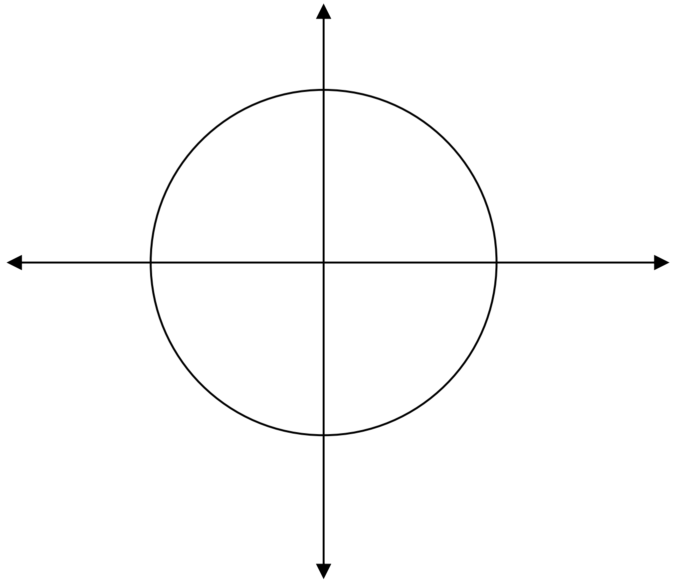
$$\csc\theta = \frac{1}{y} \quad (y \neq 0)$$

$$\cos\theta = x$$

$$\sec\theta = \frac{1}{x} \quad (x \neq 0)$$

$$\tan\theta = \frac{y}{x} \quad (x \neq 0)$$

$$\cot\theta = \frac{x}{y} \quad (y \neq 0)$$



Note: $(x, y) = (\cos\theta, \sin\theta)$

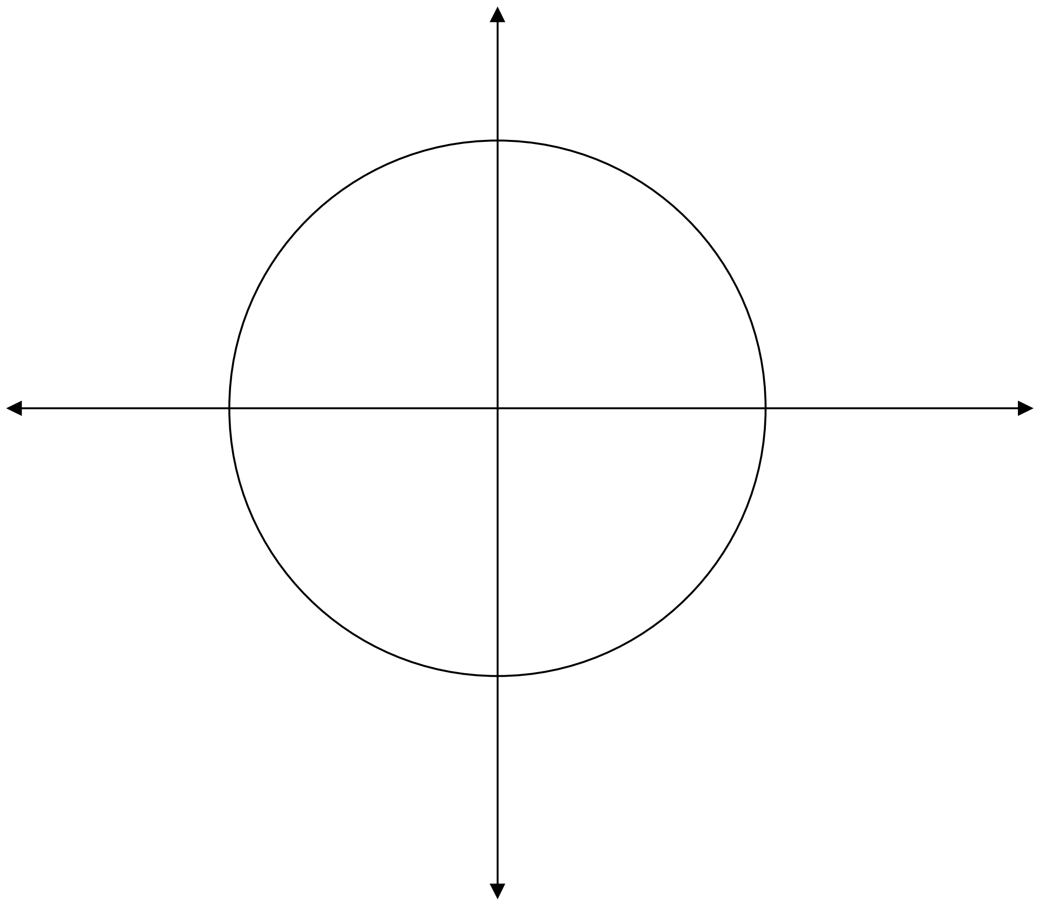
An identity is a statement that is true for all allowed values of the variable.

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\cos\theta}{\sin\theta}$$

$$\csc\theta = \frac{1}{\sin\theta}$$

$$\sec\theta = \frac{1}{\cos\theta}$$



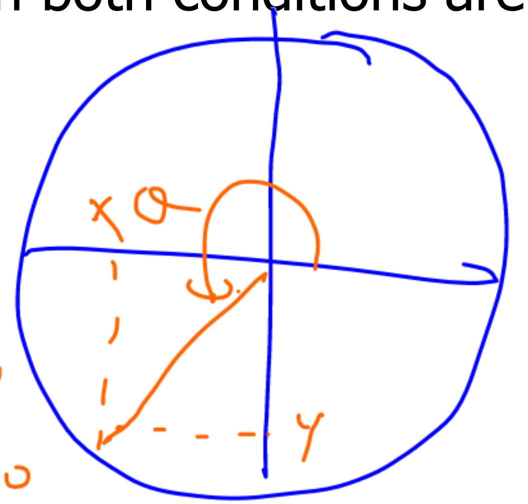
Example 4: Name the quadrant in which both conditions are true.

a. $\csc\theta < 0$ and $\tan\theta > 0$

quad III

$\tan\theta > 0$

$\csc\theta < 0$

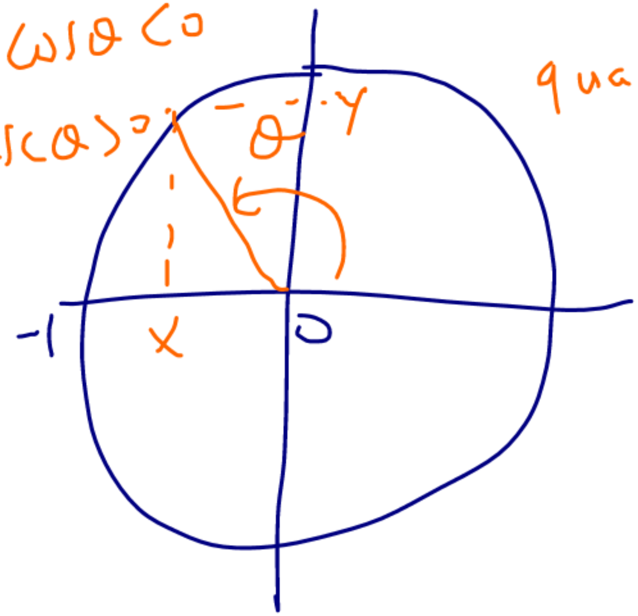


b. $\cos\theta < 0$ and $\csc\theta > 0$

$\cos\theta < 0$

$\csc\theta > 0$

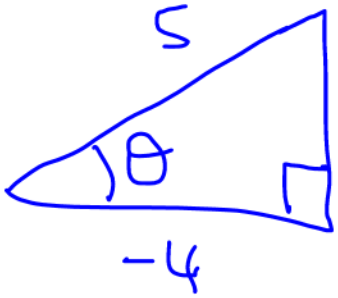
quad II

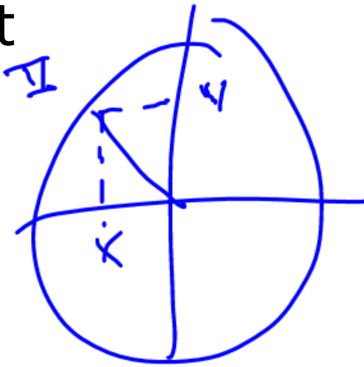


Example 5: Let the point P(x, y) denote the point where the terminal side of angle θ (in standard position) meets the unit circle. Suppose that $x = -\frac{4}{5}$ and $\frac{\pi}{2} < \theta < \pi$. Evaluate the six trig functions.

$\sin\theta > 0$ II

quad II



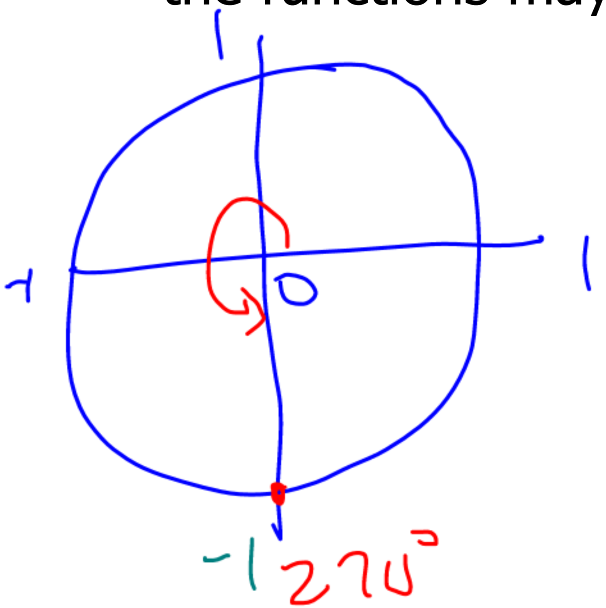


$\cos\theta = -\frac{4}{5}$; $\sec\theta = -\frac{5}{4}$

$\sin\theta = \frac{3}{5}$; $\csc = \frac{5}{3}$

$\tan\theta = -\frac{3}{4}$; $\cot\theta = -\frac{4}{3}$

Example 6: Sketch an angle measuring 270° in the coordinate plane. Then give the six trigonometric functions of the angle. Note that some of the functions may be undefined.



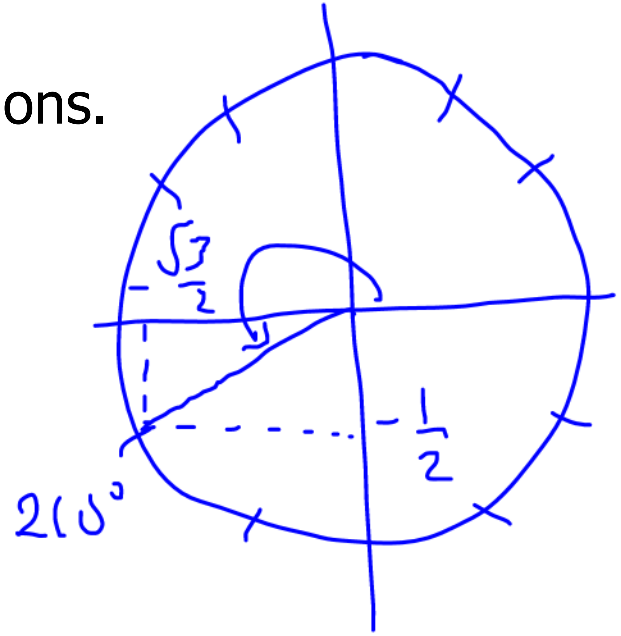
$\sin 270° = -1$; $\csc 270° = -1$

$\cos 270° = 0$; $\sec 270° = $ undef

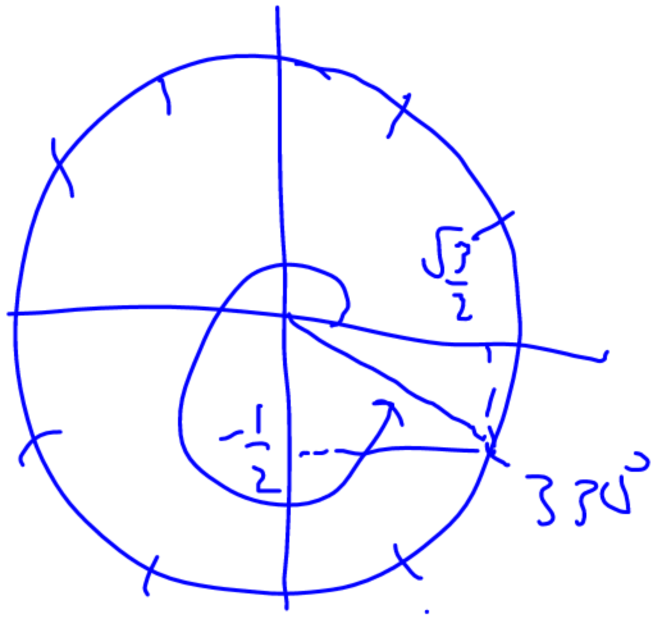
$\tan 270° = $ undef ; $\cot 270° = 0$

Example 7: Evaluate each of the following trig functions.

a. $tan\ 210° = \dfrac{-\frac{1}{2}}{-\frac{\sqrt{3}}{2}} = \dfrac{1}{\sqrt{3}} = \boxed{\dfrac{\sqrt{3}}{3}}$



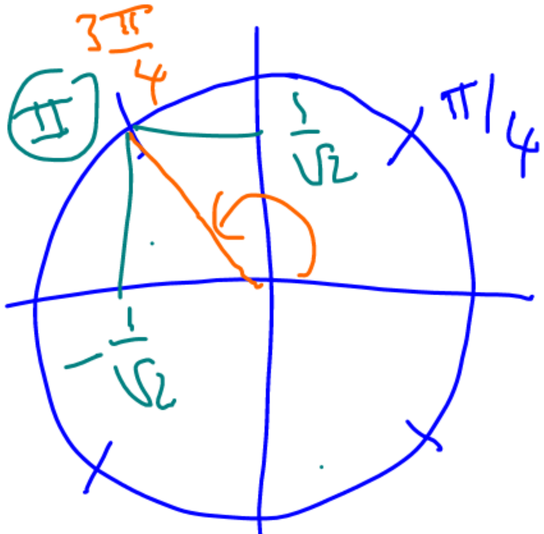
b. $csc\ 330° = \boxed{-2}$

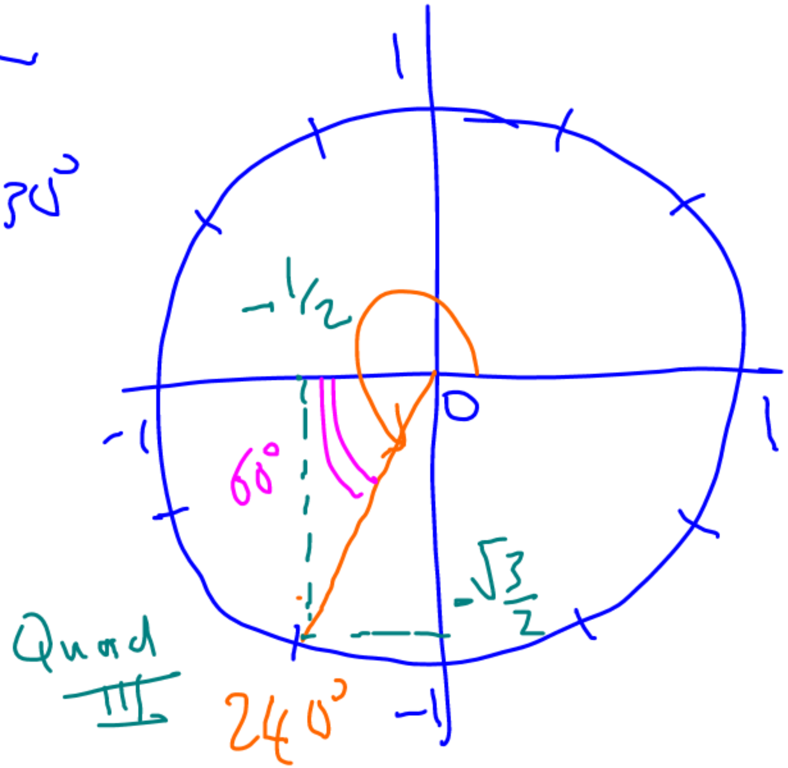


c. $sin\ 240° = \boxed{-\dfrac{\sqrt{3}}{2}}$

$= -\dfrac{\sqrt{3}}{2}$

d. $sec\dfrac{3\pi}{4} = \boxed{-\sqrt{2}}$





Learn by heart the values of the sine, cosine and tangent for each of the quadrantal angles and the special angles we have discussed -- 30°, 45°, 60° and all their multiples.

Once you learn the first quadrant, all the other values are the same, except for their signs --- positive or negative. So, in effect, you really only have 5 values to remember.

$0 \qquad \frac{1}{2} \qquad \frac{\sqrt{2}}{2} \qquad \frac{\sqrt{3}}{2} \qquad 1$

$30° = \frac{\pi}{6}$

$45° = \frac{\pi}{4}$

$60° = \frac{\pi}{3}$