ENGR 2405 Chapter 8

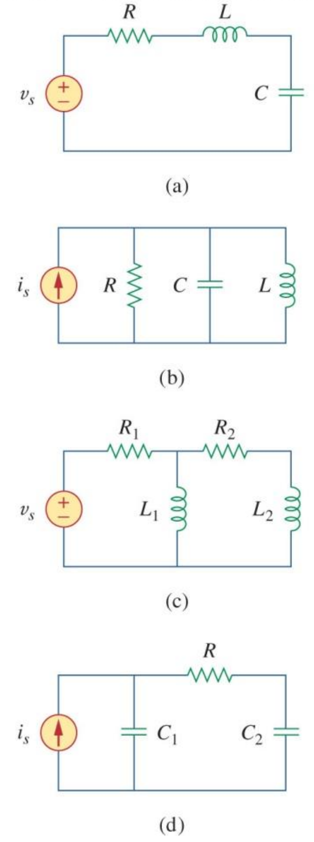
Second Order Circuits

Overview

- The previous chapter introduced the concept of first order circuits.
- This chapter will expand on that with second order circuits: those that need a second order differential equation.
- RLC series and parallel circuits will be discussed in this context.
- The step response of these circuits will be covered as well.
- Finally the concept of duality will be discussed.

Second Order Circuits

- **The previous chapter considered circuits which only required first order differential equations to solve.**
- **However, when more than one "storage element", *i.e.* capacitor or inductor is present, the equations require second order differential equations**
- **The analysis is similar to what was done with first order circuits**
- **This time, though we will only consider DC independent sources**

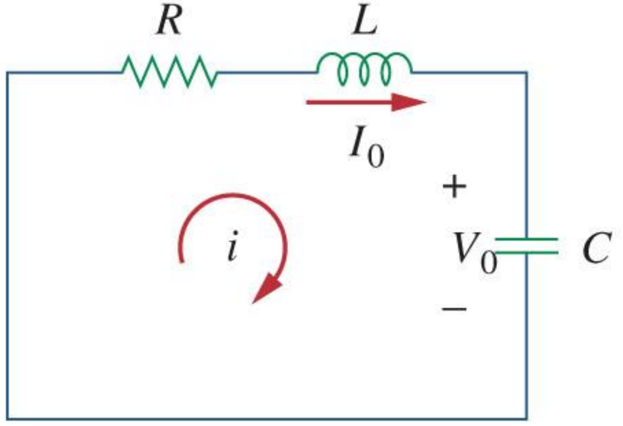


Finding Initial and Final Values

- **Working on second order system is harder than first order in terms of finding initial and final conditions.**
- **You need to know the derivatives, *dv/dt* and *di/dt* as well.**
- **Getting the polarity across a capacitor and the direction of current through an inductor is critical.**
- **Capacitor voltage and inductor current are always continuous.**

Source Free Series RLC

- **Consider the circuit shown.**
- **The energy at t=0 is stored in the capacitor, represented by V_0 and in the inductor, represented by I_0.**



$$v(0) = \frac{1}{C}\int_{-\infty}^{0} i\,dt = V_0$$

$$i(0) = I_0$$

Source Free Series RLC

- **Applying KVL around the loop:**

$$Ri + L\frac{di}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(\tau)\,d\tau = 0$$

- **The integral can be eliminated by differentiation:**

$$\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$$

- **Here you can see the second order equation that results**

Source Free Series RLC

- **Two initial conditions are needed for solving this problem.**
- **The initial current is given.**
- **The first derivative of the current can also be had:**

$$Ri(0) + L\frac{di(0)}{dt} + V_0 = 0$$

- **Or**

$$\frac{di(0)}{dt} = -\frac{1}{L}\left(RI_0 + V_0\right)$$

Source Free Series RLC

- **Based on the first order solutions, we can expect that the solution will be in exponential form.**
- **The equation will then be:**

$$Ae^{st}\left(s^2+\frac{R}{L}s+\frac{1}{LC}\right)=0$$

- **For which the solutions are:**

$$s_1=-\alpha+\sqrt{\alpha^2-\omega_o^2}\qquad s_2=-\alpha-\sqrt{\alpha^2-\omega_o^2}$$

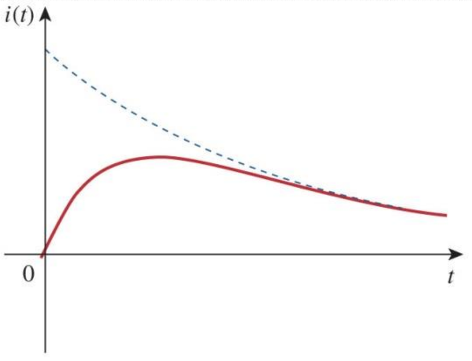
$$\alpha=\frac{R}{2L}\qquad \omega_0=\frac{1}{\sqrt{LC}}$$

Overdamped ($\alpha>\omega_0$)

- **When $\alpha>\omega_0$, the system is overdamped**
- **In this case, both s_1 and s_2 are real and negative.**
- **The response of the system is:**

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- **From this, we should not expect to see an oscillation**



(a)

Critically Damped ($\alpha=\omega_0$)

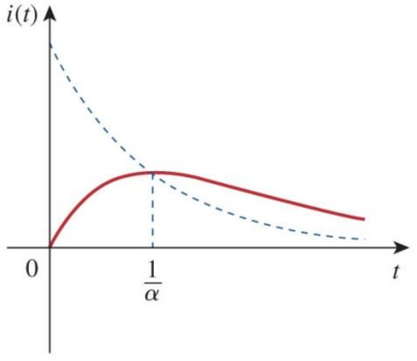
- When $\alpha=\omega_0$, the system is critically damped.
- The differential equation becomes:

$$\frac{d}{dt}\left(e^{\alpha t} i\right) = A_1$$

- For which the solution is:

$$i(t) = \left(A_2 + A_1 t\right) e^{-\alpha t}$$

- There are two components to the response, an exponential decay and an exponential decay multiplied by a linear term



(b)

Underdamped ($\alpha<\omega_0$)

- **When $\alpha<\omega_0$, the system is considered to be underdamped**
- **In this case, the solution will be:**

$$i(t) = e^{-\alpha t}\left(B_1 \cos \omega_d t + B_2 \sin \omega_d t\right)$$

- **Where** $j = \sqrt{-1}$ **and** $\omega_d = \sqrt{\omega_0^2 - \alpha^2}$
- **ω_0 is often called the undamped natural frequency**
- **ω_d is called the damped natural frequency**

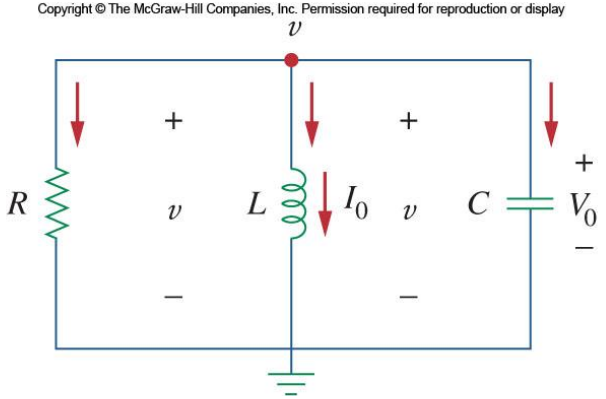
Damping and RLC networks

- **RLC networks can be charaterized by the following:**

1. **The behavior of these networks is captured by the idea of damping**
2. **Oscillatory response is possible due to the presence of two types of energy storage elements.**
3. **It is typically difficult to tell the difference between damped and critically damped responses.**

Source Free Parallel RLC Network

- **Now let us look at parallel forms of RLC networks**
- **Consider the circuit shown**
- **Assume the initial current and voltage to be:**



$$i(0) = I_0 = \frac{1}{L}\int_{-\infty}^{0} v(t)\,dt$$

$$v(0) = V_0$$

Source Free Parallel RLC Network

- **Applying KCL to the top node we get:**

$$\frac{v}{R} + \frac{1}{L}\int_{-\infty}^{t} v(\tau)\,d\tau + C\frac{dv}{dt} = 0$$

- **Taking the derivative with respect to t gives:**

$$\frac{d^2v}{dt^2} + \frac{1}{RC}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

- **The characteristic equation for this is:**

$$s^2 + \frac{1}{RC}s + \frac{1}{LC} = 0$$

Source Free Parallel RLC Network

- **From this, we can find the roots of the characteristic equation to be:**

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

- **As in last time, there are three scenarios to consider.**

Damping

- **For the overdamped case, the roots are real and negetive, so the response is:**

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- **For critically damped, the roots are real and equal, so the response is:**

$$v(t) = (A_2 + A_1 t) e^{-\alpha t}$$

Underdamped

- **In the underdamped case, the roots are complex and so the response will be:**

$$v(t) = e^{-\alpha t}\left(A_1 \cos \omega_d t + A_2 \sin \omega_d t\right)$$

- **To get the values for the constants, we need to know *v(0)* and *dv(0)/dt.***
- **To find the second term, we use:**

$$\frac{V_0}{R} + I_0 + C\frac{dv(0)}{dt} = 0$$

Underdamped

- The voltage waveforms will be similar to those shown for the series network.
- Note that in the series network, we first found the inductor current and then solved for the rest from that.
- Here we start with the capacitor voltage and similarly, solve for the other variables from that.

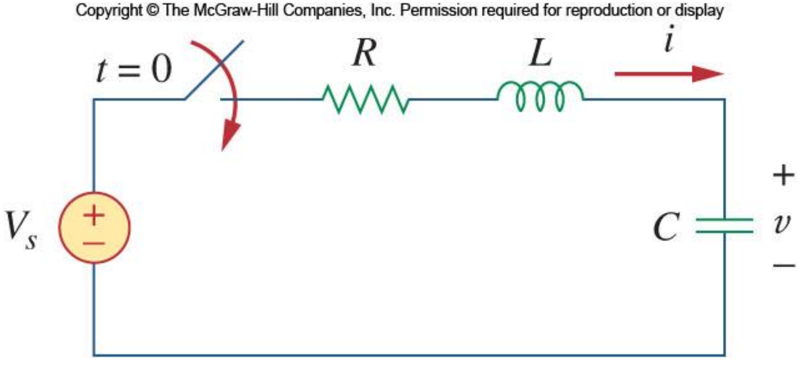
Step Response of a Series RLC Circuit

- **Now let us consider what happens when a DC voltage is suddenly applied to a second order circuit.**
- **Consider the circuit shown. The switch closes at t=0.**
- **Applying KVL around the loop for t>0:**

$$L\frac{di}{dt}+Ri+v=V_s$$

- **but**

$$i=C\frac{dv}{dt}$$



Step Response of a Series RLC Circuit

- **Substituting for *i* gives:**

$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{v}{LC}=\frac{V_s}{LC}$$

- **This is similar to the response for the source free version of the series circuit, except the variable is different.**
- **The solution to this equation is a combination of transient response and steady state**

$$v(t)=v_t(t)+v_{ss}(t)$$

Step Response of a Series RLC Circuit

- **The transient response is in the same form as the solutions for the source free version.**
- **The steady state response is the final value of *v(t)*. In this case, the capacitor voltage will equal the source voltage.**

Step Response of a Series RLC Circuit

- **The complete solutions for the three conditions of damping are:**

$$v(t) = V_s + A_1 e^{s_1 t} + A_2 e^{s_2 t} \quad \text{(Overdamped)}$$

$$v(t) = V_s + (A_1 + A_2) e^{-\alpha t} \quad \text{(Critically Damped)}$$

$$v(t) = V_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t) \quad \text{(Underdamped)}$$

- **The variables A_1 and A_2 are obtained from the initial conditions, $v(0)$ and $dv(0)/dt$.**

Step Response of a Parallel RLC Circuit

- **The same treatment given to the parallel RLC circuit yields the same result.**
- **The response is a combination of transient and steady state responses:**

$$i(t) = I_s + A_1 e^{\tau_1 t} + A_2 e^{\tau_2 t} \quad \text{(Overdamped)}$$

$$i(t) = I_s + (A_1 + A_2 t) e^{-\alpha t} \quad \text{(Critally Damped)}$$

$$i(t) = I_s + (A_1 \cos \omega_d t + A_2 \sin \omega_d t) e^{-\alpha t} \quad \text{(Underdamped)}$$

- **Here the variables A_1 and A_2 are obtained from the initial conditions, *i(0)* and *di(0)/dt*.**

General Second Order Circuits

- **The principles of the approach to solving the series and parallel forms of RLC circuits can be applied to second order circuits in general:**
- **The following four steps need to be taken:**

1. **First determine the initial conditions, *x(0)* and *dx(0)/dt*.**

Second Order Op-amp Circuits

2. **Turn off the independent sources and find the form of the transient response by applying KVL and KCL.**

- **Depending on the damping found, the unknown constants will be found.**

3. **We obtain the stead state response as:**

$$x_{ss}(t) = x(\infty)$$

Where *x(∞)* is the final value of *x* obtained in step 1

Second Order Op-amp Circuits II

4. **The total response is now found as the sum of the transient response and steady-state response.**

$$x(t) = x_t(t) + x_{ss}(t)$$

Duality

- The concept of duality is a time saving measure for solving circuit problems.
- It is based on the idea that circuits that appear to be different may be related to each other.
- They may use the same equations, but the roles of certain complimentary elements are interchanged.
- The following is a table of dual pairs

Duality

TABLE 8.1

Dual pairs.

Resistance R	Conductance G
Inductance L	Capacitance C
Voltage v	Current i
Voltage source	Current source
Node	Mesh
Series path	Parallel path
Open circuit	Short circuit
KVL	KCL
Thevenin	Norton

Duality

- **Once you know the solution to one circuit, you have the solution to the dual circuit.**
- **Finding the dual of a circuit can be done with a graphical method:**

1. **Place a node at the center of each mesh of a given circuit. Place the reference node outside the given circuit.**
2. **Draw lines between the nodes such that each line crosses an element. Replace the element with its dual**

Duality

3. To determine the polarity of voltage sources and of current sources, follow this rule: A voltage source that produces a positive (clockwise) mesh current has as its dual a current source whose reference direction is from the ground to the nonreference node.

- When in doubt, one can refer to the mesh or nodal equations of the dual circuit.