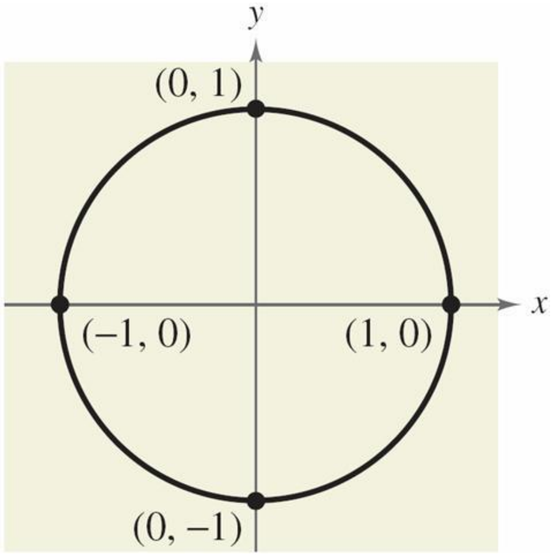
# 1.2 TRIGONOMETRIC FUNCTIONS: THE UNIT CIRCLE

# The Unit Circle

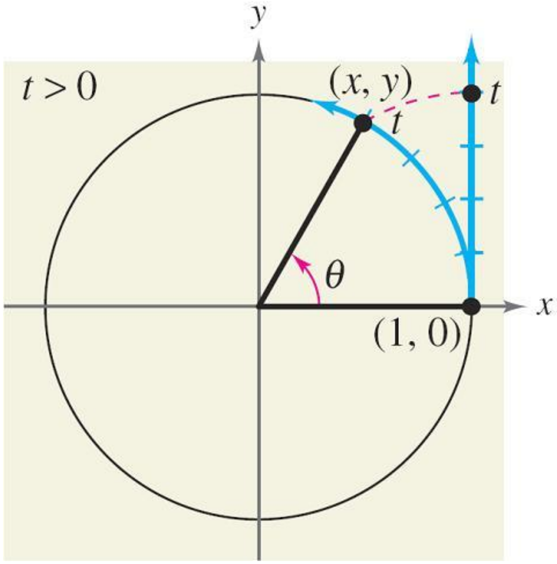
Consider the **unit circle** given by

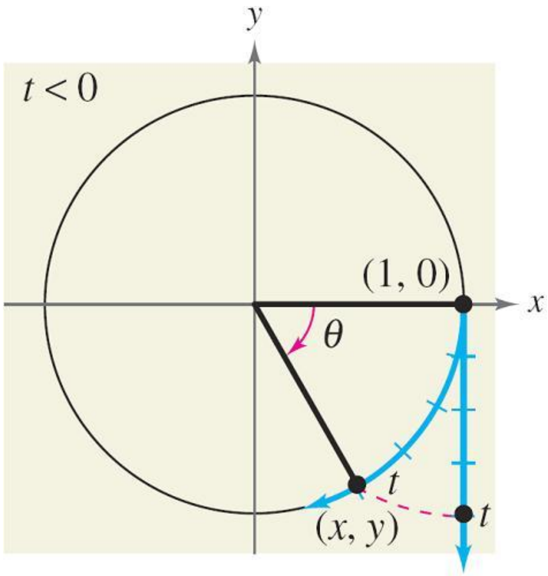
$$x^2 + y^2 = 1$$



# The Unit Circle

Imagine that the real number line is wrapped around this circle, with positive numbers corresponding to a counterclockwise wrapping and negative numbers corresponding to a clockwise wrapping.



# The Unit Circle

Each real number $t$ corresponds to a point $(x, y)$ on the circle.

For example, the real number 0 corresponds to the point (1, 0).

Moreover, because the unit circle has a circumference of $2\pi$, the real number $2\pi$ also corresponds to the point (1, 0).

# The Trigonometric Functions

**sine cosecant cosine secant tangent cotangent**

These six functions are normally abbreviated sin, csc, cos, sec, tan, and cot, respectively.

# The Trigonometric Functions

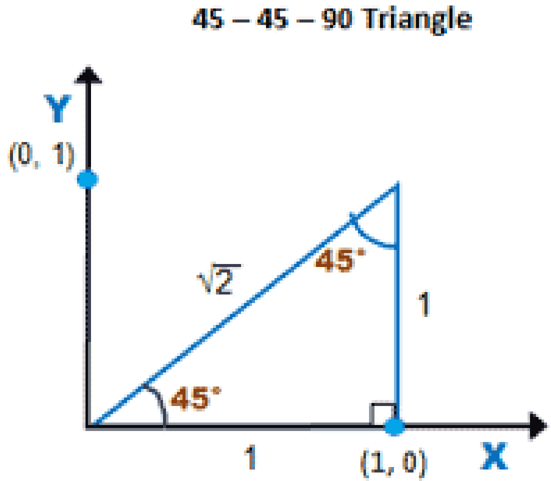
## Definitions of Trigonometric Functions

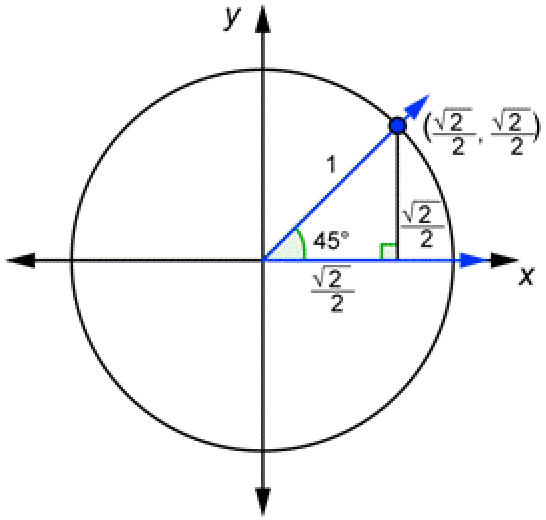
Let $t$ be a real number and let $(x, y)$ be the point on the unit circle corresponding to $t$.

$$\sin t = y \qquad \cos t = x \qquad \tan t = \frac{y}{x}, \quad x \neq 0$$

$$\csc t = \frac{1}{y}, \quad y \neq 0 \qquad \sec t = \frac{1}{x}, \quad x \neq 0 \qquad \cot t = \frac{x}{y}, \quad y \neq 0$$

# 45 degree angle



# The Trigonometric Functions

The unit circle has been divided into eight equal arcs, corresponding to values of

$$0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}, \text{ and } 2\pi.$$

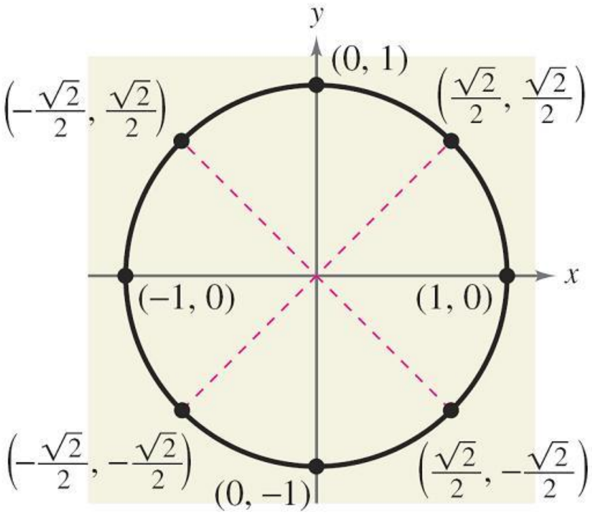
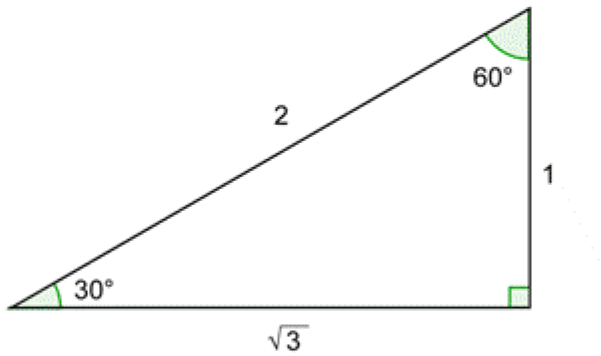
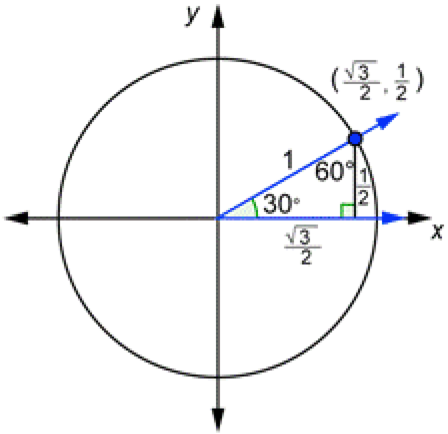



**Figure 1.22**

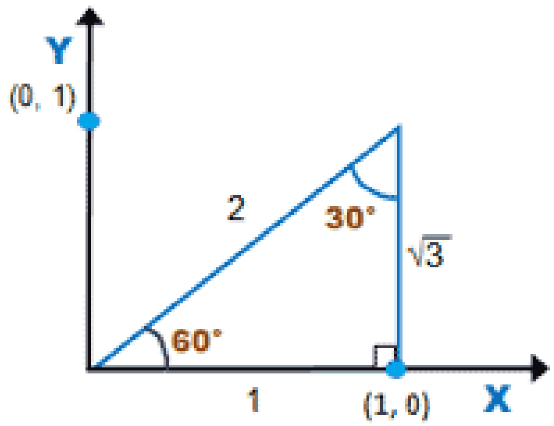


60°
2
1
30°
√3

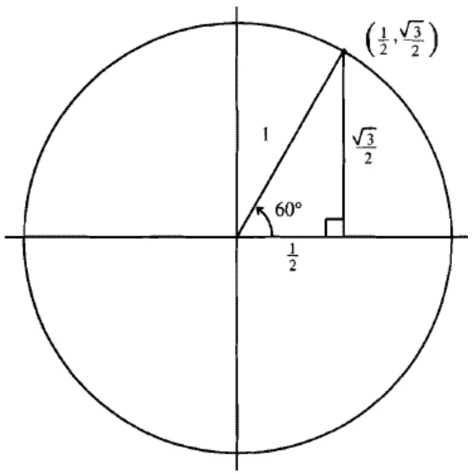



y
(√3/2, 1/2)
1
60°
1/2
30°
x
√3/2

# 60 degree



Y
(0, 1)
2
30°
$\sqrt{3}$
60°
1
(1, 0)
X

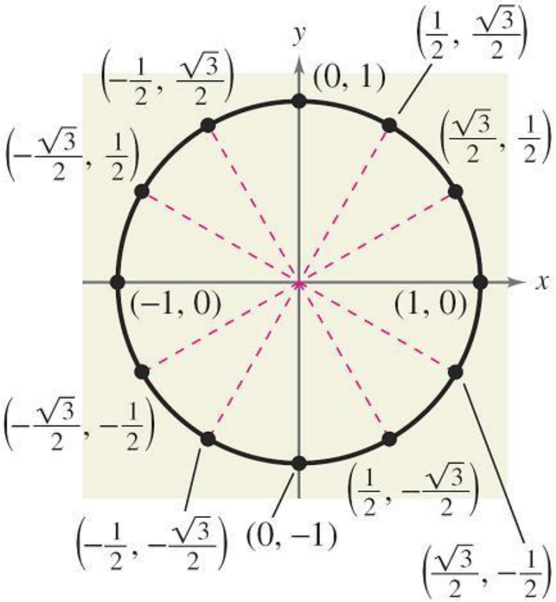



$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
1
$\frac{\sqrt{3}}{2}$
60°
$\frac{1}{2}$

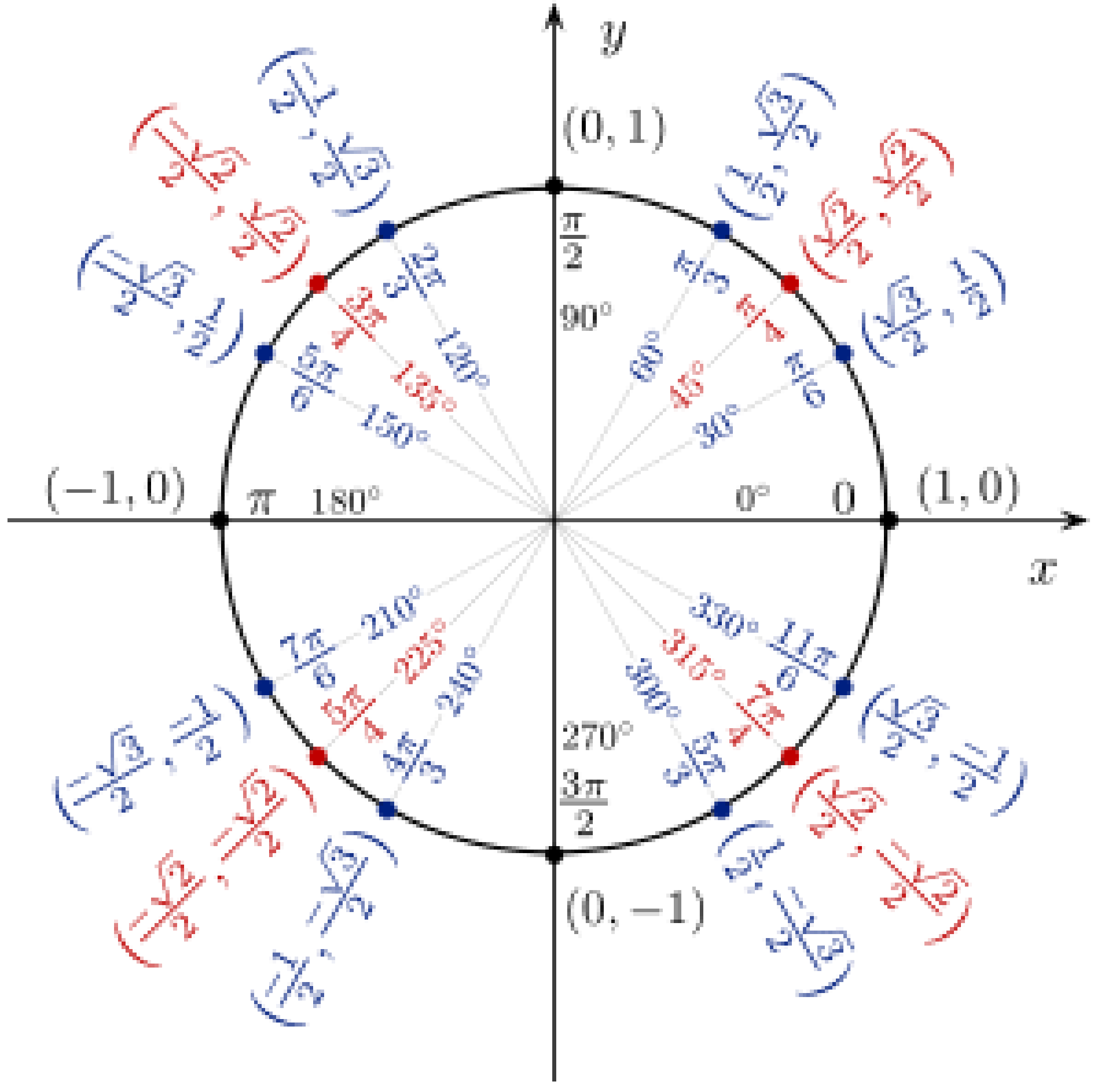
# The Trigonometric Functions

The unit circle has been divided into 12 equal arcs, corresponding to *t*-values of

$$0, \frac{\pi}{6}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}, \frac{5\pi}{6}, \pi, \frac{7\pi}{6}, \frac{4\pi}{3}, \frac{3\pi}{2}, \frac{5\pi}{3}, \frac{11\pi}{6}, \text{ and } 2\pi.$$



# Unit Circle



y
x
(0, 1)
(1, 0)
(−1, 0)
(0, −1)
$\frac{\pi}{2}$
90°
$\frac{\pi}{3}$
60°
$\frac{\pi}{4}$
45°
$\frac{\pi}{6}$
30°
0°
0
$\frac{2\pi}{3}$
120°
$\frac{3\pi}{4}$
135°
$\frac{5\pi}{6}$
150°
$\pi$
180°
$\frac{7\pi}{6}$
210°
$\frac{5\pi}{4}$
225°
$\frac{4\pi}{3}$
240°
270°
$\frac{3\pi}{2}$
$\frac{5\pi}{3}$
300°
$\frac{7\pi}{4}$
315°
$\frac{11\pi}{6}$
330°
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(\frac{-\sqrt{3}}{2}, \frac{-1}{2}\right)$
$\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$
$\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$

# Example 1 – *Evaluating Trigonometric Functions*

Evaluate the six trigonometric functions at each real number.

**a.** $t = \frac{\pi}{6}$ **b.** $t = \frac{5\pi}{4}$ **c.** $t = \pi$

Solution:

Begin by finding the corresponding point (*x*, *y*) on the unit circle. Then use the definitions of trigonometric functions.

**a.** $t = \frac{\pi}{6}$ corresponds to the point $(x, y) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

# Example 1 – *Solution*

cont'd

$$\sin\frac{\pi}{6} = y = \frac{1}{2}$$

$$\cos\frac{\pi}{6} = x = \frac{\sqrt{3}}{2}$$

$$\tan\frac{\pi}{6} = \frac{y}{x} = \frac{1/2}{\sqrt{3}/2} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$$

$$\csc\frac{\pi}{6} = \frac{1}{y} = \frac{1}{1/2} = 2$$

# Example 1 – *Solution*

cont'd

$$\sec\frac{\pi}{6} = \frac{1}{x} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$$

$$\cot\frac{\pi}{6} = \frac{x}{y} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

# Example 1 – *Solution*

cont'd

**b.** $t = \frac{5\pi}{4}$ corresponds to the point ($x$, $y$) = $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$.

$$\sin\frac{5\pi}{4} = y = -\frac{\sqrt{2}}{2}$$

$$\cos\frac{5\pi}{4} = x = -\frac{\sqrt{2}}{2}$$

$$\tan\frac{5\pi}{4} = \frac{y}{x} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

$$\csc\frac{5\pi}{4} = \frac{1}{y} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

# Example 1 – *Solution*

cont'd

$$\sec\frac{5\pi}{4} = \frac{1}{x} = -\frac{2}{\sqrt{2}} = -\sqrt{2}$$

$$\cot\frac{5\pi}{4} = \frac{x}{y} = \frac{-\sqrt{2}/2}{-\sqrt{2}/2} = 1$$

# Example 1 – *Solution*

cont'd

**c.** $t = \pi$ corresponds to the point $(x, y) = (-1, 0)$.

$$\sin \pi = y = 0$$

$$\cos \pi = x = -1$$

$$\tan \pi = \frac{y}{x} = \frac{0}{-1} = 0$$

# Example 1 – *Solution*

cont'd

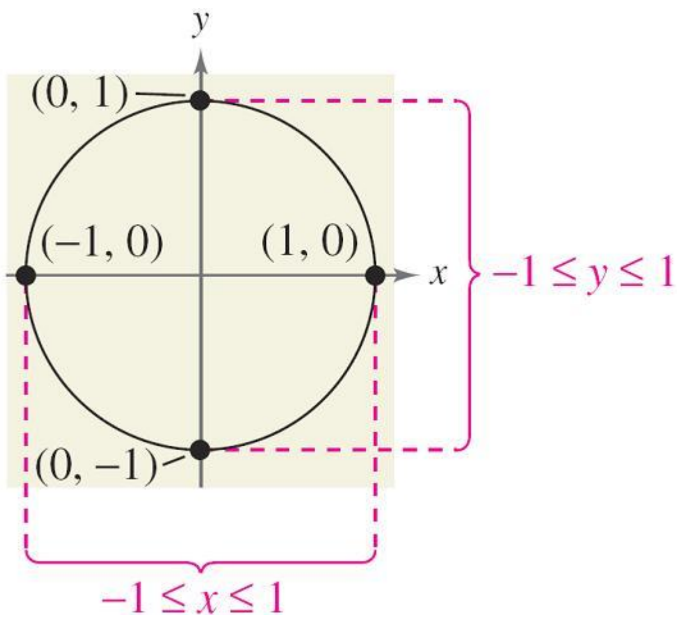
$\csc \pi = \frac{1}{y}$ is undefined.

$$\sec \pi = \frac{1}{x} = \frac{1}{-1} = -1$$

$\cot \pi = \frac{x}{y}$ is undefined.

# Domain and Period of Sine and Cosine

The *domain* of the sine and cosine functions is the set of all real numbers. To determine the *range* of these two functions, consider the unit circle.



# Domain and Period of Sine and Cosine

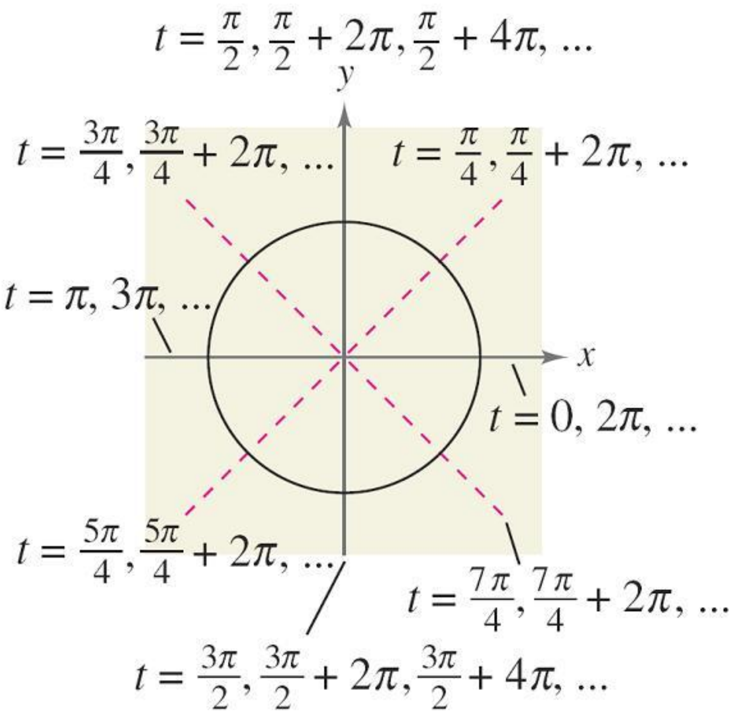
We associate sin t with y and cost t with x.

$$-1 \le y \le 1 \quad \text{and} \quad -1 \le x \le 1$$

$$-1 \le \sin t \le 1 \qquad\qquad -1 \le \cos t \le 1$$

# Domain and Period of Sine and Cosine

Adding $2\pi$ to each value of *t* in the interval $[0, 2\pi]$ completes a second revolution around the unit circle.



# Domain and Period of Sine and Cosine

$$\sin(t + 2n\pi) = \sin t$$

and

$$\cos(t + 2n\pi) = \cos t$$

for any integer *n* and real number *t*. Functions that behave in such a repetitive (or cyclic) manner are called **periodic.**

# Domain and Period of Sine and Cosine

Recall that a function $f$ is *even* if $f(-t) = f(t)$, and is *odd* if $f(-t) = -f(t)$.

**Even and Odd Trigonometric Functions**

The cosine and secant functions are *even*.

$$\cos(-t) = \cos t \qquad \sec(-t) = \sec t$$

The sine, cosecant, tangent, and cotangent functions are *odd*.

$$\sin(-t) = -\sin t \qquad \csc(-t) = -\csc t$$

$$\tan(-t) = -\tan t \qquad \cot(-t) = -\cot t$$

# Example 3 – *Using the Period to Evaluate the Sine and Cosine*

**a.** Because $\frac{13\pi}{6} = 2\pi + \frac{\pi}{6}$,

$$\sin \frac{13\pi}{6} = \sin\left(2\pi + \frac{\pi}{6}\right)$$

$$= \sin \frac{\pi}{6} \qquad = \frac{1}{2}.$$

**b.** Because $-\frac{7\pi}{2} = -4\pi + \frac{\pi}{2}$, you have

$$\cos\left(-\frac{7\pi}{2}\right) = \cos\left(-4\pi + \frac{\pi}{2}\right)$$

$$= \cos \frac{\pi}{2} \qquad = 0.$$

# Example 3 – *Using the Period to Evaluate the Sine and Cosine*

cont'd

**c.** If $\sin t = \frac{4}{5}$ , what is $sin(-t)$?

Since the sine function is odd we get

$$sin(-t) = -\frac{4}{5}$$

# Evaluating Trigonometric Functions with a Calculator

# Evaluating Trigonometric Functions with a Calculator

When evaluating a trigonometric function with a calculator, you need to set the calculator to the desired *mode* of measurement (*degree* or *radian*).

Most calculators do not have keys for the cosecant, secant, and cotangent functions. To evaluate these functions, you can use the $x^{-1}$ key with their respective reciprocal functions sine, cosine, and tangent.

# Evaluating Trigonometric Functions with a Calculator

For instance, to evaluate csc($\pi$/8), use the fact that

$$\csc\frac{\pi}{8} = \frac{1}{\sin(\pi/8)}$$

and enter the following keystroke sequence in *radian* mode.

( SIN ( $\pi$ ÷ 8 ) ) $x^{-1}$ ENTER    Display 2.6131259

# Example 4 – *Using a Calculator*

| *Function* | *Mode* | *Calculator Keystrokes* | *Display* |
|---|---|---|---|
| **a.** $\sin \frac{2\pi}{3}$ | Radian | [SIN] [(] 2 [$\pi$] [÷] 3 [)] [ENTER] | 0.8660254 |
| **b.** cot 1.5 | Radian | [(] [TAN] [(] 1.5 [)] [)] [$x^{-1}$] [ENTER] | 0.0709148 |